Midterm Exam No. 01 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Date: 2016 Feb 19

1. (**20 points.**) The Hamiltonian for the motion of a particle of mass m in a constant gravitational field $\mathbf{g} = -g\hat{\mathbf{z}}$ is

$$H(z,p,t) = \frac{p^2}{2m} + mgz. \tag{1}$$

   (a) Show that the Hamilton equations of motion are

$$\frac{dz}{dt} = \frac{p}{m}, \tag{2a}$$

$$\frac{dp}{dt} = -mg. \tag{2b}$$

   (b) Show that the Hamilton-Jacobi equation

$$-\frac{\partial W}{\partial t} = H\left(z, \frac{\partial W}{\partial z}, t\right), \tag{3}$$

   in terms of Hamilton's principal function $W(z,t)$ is given by

$$-\frac{\partial W}{\partial t} = \frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz. \tag{4}$$

   Further, show that

$$W(z,t) = -Et - \frac{2}{3}\frac{\sqrt{2m}}{mg}(E - mgz)^{\frac{3}{2}} \tag{5}$$

   is a solution to the Hamilton-Jacobi equation up to a constant.

   (c) Hamilton's principal function allows us to identify canonical transformations $Q = Q(z,p,t)$ and $P = P(z,p,t)$, such that

$$\frac{\partial W}{\partial q} = p, \qquad \frac{\partial W}{\partial Q} = -P, \qquad \frac{\partial W}{\partial t} = -H, \tag{6a}$$

$$\frac{\partial W}{\partial p} = 0, \qquad \frac{\partial W}{\partial P} = 0, \tag{6b}$$

   with the feature that the new coordinates are constants of motion,

$$\frac{dQ}{dt} = 0 \qquad \text{and} \qquad \frac{dP}{dt} = 0. \tag{7}$$

To this end, choose $Q = E$ and then evaluate

$$P = -\frac{\partial W}{\partial Q} = t + \frac{p}{mg}. \tag{8}$$

Hint: Use $p = \frac{\partial W}{\partial z}$.

(d) Show that

$$Q = \frac{p^2}{2m} + mgz, \tag{9a}$$
$$P = t + \frac{p}{mg}, \tag{9b}$$

is a canonical transformation. That is, show that $[Q, P]^{\text{P.B.}}_{q,p} = 1$. Further, verify that

$$\frac{dQ}{dt} = 0, \tag{10a}$$
$$\frac{dP}{dt} = 0, \tag{10b}$$
$$K(Q, P, t) = H(z, p, t) + \frac{\partial W}{\partial t} = 0. \tag{10c}$$

2. (**20 points.**) Consider the (time independent) Hamiltonian

$$H = H(x, p), \tag{11}$$

which satisfies the equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \tag{12}$$

Recollect that the Lagrangian, which will temporarily be called the x-Lagrangian here, is defined by the construction

$$L_x = p\frac{dx}{dt} - H \tag{13}$$

and implies the equations of motion

$$p = \frac{\partial L_x}{\partial \left(\frac{dx}{dt}\right)}, \qquad \frac{dp}{dt} = \frac{\partial L_x}{\partial x}. \tag{14}$$

Now, define the p-Lagrangian using the construction

$$L_p = -x\frac{dp}{dt} - H \tag{15}$$

and derive the equations of motion satisfied by the p-Lagrangian.

Comments: The opposite sign in the construction of the p-Lagrangian is motivated by the action principle, which does not care for a total derivative. You could use a specific Hamiltonian, for example that of a harmonic oscillator, as a guide.

3. (**20 points.**) Consider the rotation matrix

$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{16}$$

   (a) Find the eigenvalues of the matrix **A**.

   (b) Find the normalized eigenvectors of matrix **A**.

   (c) Determine the matrix that diagonalizes the matrix **A**.

   (d) What can you then conclude about the eigenvalues and eigenvectors of $\mathbf{A}^{107}$? Find them.

4. (**20 points.**) Hamiltonian for a charge particle of mass m and charge q in a magnetic field **B** is given by

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2, \tag{17}$$

where

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{18}$$

Let

$$\frac{\partial \mathbf{A}}{\partial t} = 0. \tag{19}$$

Show that the Poisson braket

$$\big[\mathbf{v}, \mathbf{v}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{q}{m^2 c}\mathbf{1} \times \mathbf{B}, \tag{20}$$

where $\mathbf{v} = d\mathbf{x}/dt$.

5. (**20 points.**) The Hamiltonian for a Kepler problem (or a classical hydrogenic atom) is

$$H(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2m} - \frac{\alpha}{r}, \tag{21}$$

where $r = |\mathbf{r}|$ and $p = |\mathbf{p}|$. The Hamilton's equations of motion for the Kepler are

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{m}, \qquad \frac{d\mathbf{p}}{dt} = -\alpha\frac{\mathbf{r}}{r^3}. \tag{22}$$

The Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{m\alpha}\mathbf{p} \times \mathbf{L}, \tag{23}$$

are the three constants of motion for a Kepler problem. Under the special circumstance when $r = |\mathbf{r}|$ is also a conserved quantity, that is,

$$\frac{dr}{dt} = 0, \tag{24}$$

we have the case of circular motion. Evaluate the Laplace-Runge-Lenz vector for this case of circular orbit.