Midterm Exam No. 03 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Date: 2016 Apr 21

1. (**20 points.**) Given that $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$, are operators that transform like a vector, what can you conclude about the commutation relation of the operator

$$\mathbf{V}_1 \cdot (\mathbf{V}_2 \times \mathbf{V}_3) \tag{1}$$

with angular momentum $\mathbf{J}$? That is,

$$\big[\mathbf{V}_1 \cdot (\mathbf{V}_2 \times \mathbf{V}_3), \mathbf{J}\big] =? \tag{2}$$

2. (**20 points.**) Orbital angular momentum $\mathbf{L}$ also transforms like angular momentum $\mathbf{J}$. (The eigenvalues of the orbital angular momentum are necessarily integers, because $m = 0$ is necessarily an allowed state due to the fact that $\mathbf{r} \cdot \mathbf{L} = 0$, corresponding to the fact that rotation about $\mathbf{r}$ has no effect. This rules out half-integral values for the eigenvalues.) The orbital angular momentum has the operator construction

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{3}$$

where $\mathbf{r}$ is the position operator and $\mathbf{p}$ is the linear momentum operator which satisfies the Heisenberg uncertainty relation

$$\frac{1}{i\hbar}\big[\mathbf{r}, \mathbf{p}\big] = \mathbf{1}. \tag{4}$$

Show that

$$\mathbf{L}^\dagger = -\mathbf{p} \times \mathbf{r}. \tag{5}$$

Evaluate

$$\mathbf{L} - \mathbf{L}^\dagger. \tag{6}$$

Is orbital angular momentum $\mathbf{L}$ self-adjoint (Hermitian)?
Caution: Use index notation to avoid pitfalls while using vector operators.

3. (**20 points.**) Using the commutation relations between the position vector $\mathbf{r}$, the linear momentum vector $\mathbf{p}$, and the orbital angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, show that

$$\frac{1}{i\hbar}\big(\mathbf{p} \times \mathbf{L} + \mathbf{L} \times \mathbf{p}\big) = a\mathbf{p}, \tag{7}$$

where a is a number. Report the numerical value for a in the above expression.

4. (**20 points.**) The components of angular momentum $\mathbf{J}$ satisfy the commutation relations

$$\frac{1}{i\hbar}\big[J_i, J_j\big] = \varepsilon_{ijk} J_k. \tag{8}$$

The general properties of angular momentum can be deduced from these commutation relations. Since $\mathbf{J}^2$ is a scalar, it commutes with angular momentum $\mathbf{J}$. Thus, the common eigenvectors of $\mathbf{J}^2$ and J_z constitute a suitable set of basis vectors for discussing a dynamical system involving only the angular momentum. Let us denote the eigenvalues of these operators by the labeling scheme $\mathbf{J}'^2 = j(j+1)\hbar^2$, and $J_z{}' = m\hbar$. Thus, we write

$$\frac{1}{\hbar^2}\mathbf{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \tag{9a}$$

$$\frac{1}{\hbar}J_z|j,m\rangle = m|j,m\rangle. \tag{9b}$$

Let us also construct (non-Hermitian) operators

$$J_\pm = J_x \pm iJ_y. \tag{10}$$

Observe that $J_\pm^\dagger = J_\mp$.

(a) Show that

$$\frac{1}{\hbar}J_z\Big\{J_+|j,m\rangle\Big\} = (m+1)\Big\{J_+|j,m\rangle\Big\}. \tag{11}$$

Thus deduce that if m is an eigenvalue of J_z, then $(m+1)$ is also an eigenvlaue of J_z. Similarly, show that

$$\frac{1}{\hbar}J_z\Big\{J_-|j,m\rangle\Big\} = (m-1)\Big\{J_-|j,m\rangle\Big\}. \tag{12}$$

Thus deduce that if m is an eigenvalue of J_z, then $(m-1)$ is also an eigenvlaue of J_z.

(b) Show that

$$J_+J_- = \mathbf{J}^2 - J_z^2 + \hbar J_z \tag{13}$$

is a Hermitian operator. A Hermitian operator has real eigenvalues. But, since $J_+J_- = J_-^\dagger J_-$, infer that it has non-negative eigenvalues. Thus, deduce that

$$j(j+1) - m(m-1) \geq 0, \tag{14}$$

and then infer

$$-j \leq m \leq j+1. \tag{15}$$

(c) Show that

$$J_-J_+ = \mathbf{J}^2 - J_z^2 - \hbar J_z \tag{16}$$

is a Hermitian operator. Thus, deduce that

$$j(j+1) - m(m+1) \geq 0, \tag{17}$$

and then infer

$$-j-1 \leq m \leq j. \tag{18}$$

(d) Using Eqs. (15) and (18) in conjunction, show that

$$-j \leq m \leq j. \tag{19}$$

Further, infer that

$$j \geq 0. \tag{20}$$

Note that $\mathbf{J}^2$ being Hermitian implies $j(j+1) \geq 0$, and does not imply that j should be non-negative.

(e) The transitions from $m = -j$ to $m = j$ happen in $n = 0, 1, 2, \ldots$ steps. Thus, infer that $2j = n$. Thus,

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \tag{21a}$$

$$m = -j, -j+1, \ldots, +j. \tag{21b}$$

(f) Repeat the above analysis starting from the labeling scheme

$$\mathbf{J}'^2 = \beta\hbar^2, \quad \text{and} \quad J_z{}' = m\hbar. \tag{22}$$

5. (**20 points.**) The angular momentum can be decomposed as

$$\mathbf{J} = \mathbf{S} + \mathbf{L}, \tag{23}$$

where $\mathbf{S}$ is the spin or internal angular momentum, and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the orbital or external angular momentum. For the case $\mathbf{S} = 0$ the eigenvalues of angular momentum are necessarily integer valued, because $\mathbf{r} \cdot \mathbf{L} = 0$. Let us denote the eigenvalues by the labeling scheme $\mathbf{L}'^2 = \hbar^2 l(l+1)$ and $L_z' = \hbar m$, such that

$$\mathbf{L}^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle, \tag{24a}$$

$$L_z |l, m\rangle = \hbar m |l, m\rangle, \tag{24b}$$

where

$$l = 0, 1, 2, \ldots, \tag{25a}$$

$$m = -l, -l+1, \ldots, l. \tag{25b}$$

The eigenvectors of orbital angular momentum are suitably realized by functions on the surface of a unit sphere, coordinated by spherical polar coordinates θ' and ϕ' or the unit vector $\hat{\mathbf{r}}'$. These wavefunctions defined using the projections

$$\langle \hat{\mathbf{r}}' | l, m \rangle = Y_{lm}(\theta', \phi') \tag{26}$$

are the spherical harmonics.

(a) Show that in the position basis, here restricted to the surface of a unit sphere, we have

$$\langle \hat{\mathbf{r}}'|\mathbf{L}|\ \rangle = \langle \hat{\mathbf{r}}'|\mathbf{r}\times\mathbf{p}|\ \rangle = \frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}')\langle \hat{\mathbf{r}}'|\ \rangle. \tag{27}$$

Using Eq. (27) in Eqs. (24) show that the differential equations for spherical harmonics are given by

$$-(\mathbf{r}'\times\boldsymbol{\nabla}')\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')Y_{lm}(\theta',\phi') = l(l+1)Y_{lm}(\theta',\phi'), \tag{28a}$$

$$\frac{1}{i}\hat{\mathbf{z}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')Y_{lm}(\theta',\phi') = mY_{lm}(\theta',\phi'). \tag{28b}$$

(b) Show that the raising and lowering operators defined using

$$L_\pm = L_x \pm iL_y, \tag{29}$$

leading to raising and lowering operations

$$L_\pm|l,m\rangle = \hbar\sqrt{(l\mp m)(l\pm m+1)}\,|l,m\pm 1\rangle, \tag{30}$$

correspond to the differential equations

$$\frac{1}{i}\Big[\hat{\mathbf{x}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\pm i\hat{\mathbf{y}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\Big]Y_{lm}(\theta',\phi') = \sqrt{(l\mp m)(l\pm m+1)}\,Y_{l,m\pm1}(\theta',\phi'). \tag{31}$$

(c) Using the differential operator in spherical polar coordinates,

$$\boldsymbol{\nabla}' = \hat{\mathbf{r}}'\frac{\partial}{\partial r'} + \hat{\boldsymbol{\theta}}'\frac{1}{r'}\frac{\partial}{\partial\theta'} + \hat{\boldsymbol{\phi}}'\frac{1}{r'\sin\theta'}\frac{\partial}{\partial\phi'}, \tag{32}$$

where

$$\hat{\mathbf{r}}' = \hat{\mathbf{x}}'\sin\theta'\cos\phi' + \hat{\mathbf{y}}'\sin\theta'\sin\phi' + \hat{\mathbf{z}}'\cos\theta', \tag{33a}$$

$$\hat{\boldsymbol{\theta}}' = \hat{\mathbf{x}}'\cos\theta'\cos\phi' + \hat{\mathbf{y}}'\cos\theta'\sin\phi' - \hat{\mathbf{z}}'\sin\theta', \tag{33b}$$

$$\hat{\boldsymbol{\phi}}' = -\hat{\mathbf{x}}'\sin\phi' + \hat{\mathbf{y}}'\cos\phi', \tag{33c}$$

show that

$$\mathbf{r}'\times\boldsymbol{\nabla}' = \hat{\boldsymbol{\phi}}'\frac{\partial}{\partial\theta'} - \hat{\boldsymbol{\theta}}'\frac{1}{\sin\theta'}\frac{\partial}{\partial\phi'} \tag{34a}$$

$$= \hat{\mathbf{x}}'\left[-\sin\phi'\frac{\partial}{\partial\theta'} - \cos\phi'\cot\theta'\frac{\partial}{\partial\phi'}\right] + \hat{\mathbf{y}}'\left[\cos\phi'\frac{\partial}{\partial\theta'} - \sin\phi'\cot\theta'\frac{\partial}{\partial\phi'}\right] + \hat{\mathbf{z}}'\frac{\partial}{\partial\phi'}. \tag{34b}$$

Thus, show the correspondence

$$L_z:\quad \hat{\mathbf{z}}'\cdot\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}') = \frac{\hbar}{i}\frac{\partial}{\partial\phi'}, \tag{35a}$$

$$L_x:\quad \hat{\mathbf{x}}'\cdot\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}') = \frac{\hbar}{i}\left[-\sin\phi'\frac{\partial}{\partial\theta'} - \cos\phi'\cot\theta'\frac{\partial}{\partial\phi'}\right], \tag{35b}$$

$$L_y:\quad \hat{\mathbf{y}}'\cdot\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}') = \frac{\hbar}{i}\left[\cos\phi'\frac{\partial}{\partial\theta'} - \sin\phi'\cot\theta'\frac{\partial}{\partial\phi'}\right]. \tag{35c}$$

Further, verify the correspondence

$$L^2: \qquad \frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}')\cdot\frac{\hbar}{i}(\mathbf{r}'\times\boldsymbol{\nabla}') = \frac{\hbar^2}{i^2}\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2}\right], \tag{36a}$$

$$L_z^2: \qquad \frac{\hbar^2}{i^2}\Big[\hat{\mathbf{z}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\Big]^2 = \frac{\hbar^2}{i^2}\frac{\partial^2}{\partial\phi'^2}, \tag{36b}$$

$$L_\pm: \quad \frac{\hbar}{i}\Big[\hat{\mathbf{x}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}') \pm i\hat{\mathbf{y}}'\cdot(\mathbf{r}'\times\boldsymbol{\nabla}')\Big] = \frac{\hbar}{i}e^{\pm i\phi}\left[\pm i\frac{\partial}{\partial\theta'} - \cot\theta'\frac{\partial}{\partial\phi'}\right], \tag{36c}$$

(d) Thus, show that the eigenfunctions of angular momentum in the position basis, the spherical harmonics, satisfy the differential equations given by

$$L_z: \qquad \frac{1}{i}\frac{\partial}{\partial\phi'}Y_{lm}(\theta',\phi') = mY_{lm}(\theta',\phi'), \tag{37a}$$

$$L^2: \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2}\right]Y_{lm}(\theta',\phi') = l(l+1)Y_{lm}(\theta',\phi'), \tag{37b}$$

$$L_\pm: \qquad \frac{i}{i}e^{\pm i\phi}\left[\pm i\frac{\partial}{\partial\theta'} - \cot\theta'\frac{\partial}{\partial\phi'}\right]Y_{lm}(\theta',\phi') = \sqrt{(l\mp m)(l\pm m+1)}Y_{l,m\pm 1}(\theta',\phi'). \tag{37c}$$

Further, verify

$$\begin{aligned} L_+L_-: \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2} - \frac{\partial^2}{\partial\phi'^2} - \frac{1}{i}\frac{\partial}{\partial\phi'}\right]Y_{lm}(\theta',\phi') \\ = \Big[l(l+1) - m(m-1)\Big]Y_{lm}(\theta',\phi'), \end{aligned} \tag{38a}$$

$$\begin{aligned} L_-L_+: \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2} - \frac{\partial^2}{\partial\phi'^2} + \frac{1}{i}\frac{\partial}{\partial\phi'}\right]Y_{lm}(\theta',\phi') \\ = \Big[l(l+1) - m(m+1)\Big]Y_{lm}(\theta',\phi'), \end{aligned} \tag{38b}$$

$$\begin{aligned} L_x^2 + L_y^2: \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \cot^2\theta'\frac{\partial^2}{\partial\phi'^2}\right]Y_{lm}(\theta',\phi') \\ = \Big[l(l+1) - m^2\Big]Y_{lm}(\theta',\phi'). \end{aligned} \tag{38c}$$