Final Exam (Spring 2016)

PHYS 530A: Quantum Mechanics II

Date: 2016 May 7

1. (**20 points.**) The momentum operator $\mathbf{p}$ is a Hermitian operator. That is, $\mathbf{p}^\dagger = \mathbf{p}$. When the system is described by the state vector $|\ \rangle$, the expectation value of the momentum operator is given by
$$\mathbf{u_p} = \langle\ |\mathbf{p}|\ \rangle. \tag{1}$$
In the position-basis we have
$$\mathbf{u_p} = \int d^3x' \langle\ |\mathbf{x}'\rangle\langle\mathbf{x}'|\mathbf{p}|\ \rangle = \int d^3x' \psi^*(\mathbf{x}')\frac{\hbar}{i}\boldsymbol{\nabla}'\psi(\mathbf{x}'), \tag{2}$$
where the wavefunction $\psi(\mathbf{x}') = \langle\ |\mathbf{x}'\rangle$ is the projection of the state vector in the position-basis. Using the position-basis representation of $\mathbf{u_p}$ given by the second equality in Eq. (2) show that
$$\mathbf{u_p}^* = \mathbf{u_p}. \tag{3}$$
(Hint: Use integration by parts.)

   The orbital angular momentum is given by the operator construction
$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{4}$$
in terms of the position operator $\mathbf{r}$ and the momentum operator $\mathbf{p}$, each being Hermitian. Show that orbital angular momentum is Hermitian. That is, $\mathbf{L}^\dagger = \mathbf{L}$. The expectation value of the angular momentum, in the position-basis representation, is given by
$$\mathbf{u_L} = \int d^3x' \psi^*(\mathbf{x}')\frac{\hbar}{i}(\mathbf{x}' \times \boldsymbol{\nabla}')\psi(\mathbf{x}'). \tag{5}$$
Show that
$$\mathbf{u_L}^* = \mathbf{u_L}. \tag{6}$$

2. (**20 points.**) Using the Baker-Campbell-Hausdorff formula,
$$e^{-A}Be^{A} = B + [B,A] + \frac{1}{2!}[[B,A],A] + \frac{1}{3!}[[[B,A],A],A] + \ldots \tag{7}$$
evaluate
$$e^{-\frac{i}{\hbar}\phi J_z} J_x e^{\frac{i}{\hbar}\phi J_z}, \tag{8}$$
where J_x, J_y, and J_z, are components of the angular momentum vector (operator), and ϕ is a number representing an angle of rotation.

3. (**20 points.**) Construct the total angular momentum state $|68, -67\rangle$ for the composite system built out of two angular momenta $j_1 = 31, j_2 = 37$.

4. (**20 points.**) Using commutation relations between position $\mathbf{r}$, linear momentum $\mathbf{p}$, and angular momentum $\mathbf{L}$, verify the relation

$$\mathbf{p} \times \mathbf{L} \cdot \mathbf{p} = c\, i\hbar\, p^2, \tag{9}$$

where c is a number. Determine c.