Homework No. 01 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Due date: Wednesday, 2016 Jan 27, 4.30pm

1. (**30 points.**) The motion of a particle of mass m undergoing simple harmonic motion is described by

$$\frac{d}{dt}(mv) = -kx, \tag{1}$$

where $v = dx/dt$ is the velocity in the x direction.

   (a) Find the Lagrangian for this system that implies the equation of motion of Eq. (1) using Hamilton's principle of stationary action.

   (b) Determine the canonical momentum for this system

   (c) Determine the Hamilton $H(p, x)$ for this system.

2. (**10 points.**) The Hamiltonian is defined by the relation

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t). \tag{2}$$

Show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{3}$$

Under what circumstances is H interpreted as the energy of the system?

3. (**30 points.**) A relativistic charged particle of charge q and mass m in the presence of a known electric and magnetic field is described by

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{B}. \tag{4}$$

   (a) Find the Lagrangian for this system that implies the equation of motion of Eq. (4) using Hamilton's principle of stationary action.

   (b) Determine the canonical momentum for this system

   (c) Determine the Hamilton $H(\mathbf{p}, \mathbf{r})$ for this system.

4. (**30 points.**) Consider the Lagrangian

$$L = \frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 - V(\mathbf{r}, t). \tag{5}$$

(a) Show that principle of stationary action with respect to $\delta\mathbf{r}$ implies Newton's second law

$$m\frac{d^2\mathbf{r}}{dt^2} = -\boldsymbol{\nabla}V. \tag{6}$$

(b) Show that principle of stationary action with respect to δt implies

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 + V\right] = \frac{\partial V}{\partial t}, \tag{7}$$

which for a static potential, $\partial V/\partial t = 0$, is the statement of conservation of energy.

(c) Show that the invariance of the total time derivative term, that gets contributions only from the end points, under an infinitesimal rigid rotation

$$\mathbf{r}' = \mathbf{r} - \delta\mathbf{r}, \qquad \delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}, \tag{8}$$

implies the conservation of total angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

5. (**30 points.**) The Hamiltonian for a hydrogenic atom is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{Ze^2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{9}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two constituent particles of masses m_1 and m_2 and charges e and Ze.

(a) Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2}, \tag{10}$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{Ze^2}{r}, \tag{11}$$

where

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{12}$$

(b) Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -Ze^2\frac{\mathbf{r}}{r^3}. \tag{13}$$

(c) Verify that the Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Ze^2}\mathbf{p} \times \mathbf{L}, \tag{14}$$

are the three constants of motion for a hydrogenic atom.