Homework No. 02 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2016 Feb 9, 4.30pm

1. (**40 points.**) For two functions

$$A = A(\mathbf{x}, \mathbf{p}, t), \tag{1}$$

$$B = B(\mathbf{x}, \mathbf{p}, t), \tag{2}$$

the Poisson braket with respect to the canonical variables $\mathbf{x}$ and $\mathbf{p}$ is defined as

$$\big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \equiv \frac{\partial A}{\partial \mathbf{x}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{x}}. \tag{3}$$

Show that the Poisson braket satisfies the conditions for a Lie algebra. That is, show that

(a) Antisymmetry:

$$\big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\big[B, A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{4}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[aA + bB, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = a\big[A, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + b\big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{5}$$

Further show that

$$\big[AB, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A\big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B. \tag{6}$$

(c) Jacobi's identity:

$$\Big[A, \big[B, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \Big[B, \big[C, A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \Big[C, \big[A, B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{7}$$

2. (**40 points.**) Show that the commutator of two matrices,

$$\big[\mathbf{A}, \mathbf{B}\big] \equiv \mathbf{AB} - \mathbf{BA}, \tag{8}$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$\big[\mathbf{A}, \mathbf{B}\big] = -\big[\mathbf{B}, \mathbf{A}\big]. \tag{9}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[a\mathbf{A} + b\mathbf{B}, \mathbf{C}\big] = a\big[\mathbf{A}, \mathbf{C}\big] + b\big[\mathbf{B}, \mathbf{C}\big]. \tag{10}$$

Further show that

$$\big[\mathbf{AB}, \mathbf{C}\big] = \mathbf{A}\big[\mathbf{B}, \mathbf{C}\big] + \big[\mathbf{A}, \mathbf{C}\big]\mathbf{B}. \tag{11}$$

(c) Jacobi's identity:

$$\big[\mathbf{A}, \big[\mathbf{B}, \mathbf{C}\big]\big] + \big[\mathbf{B}, \big[\mathbf{C}, \mathbf{A}\big]\big] + \big[\mathbf{C}, \big[\mathbf{A}, \mathbf{B}\big]\big] = 0. \tag{12}$$

3. (**40 points.**) Show that the vector product of two vectors, in this problem denoted using

$$\big[\mathbf{A}, \mathbf{B}\big]_v \equiv \mathbf{A} \times \mathbf{B}, \tag{13}$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$\big[\mathbf{A}, \mathbf{B}\big]_v = -\big[\mathbf{B}, \mathbf{A}\big]_v. \tag{14}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[a\mathbf{A} + b\mathbf{B}, \mathbf{C}\big]_v = a\big[\mathbf{A}, \mathbf{C}\big]_v + b\big[\mathbf{B}, \mathbf{C}\big]_v. \tag{15}$$

Further show that

$$\big[\mathbf{A} \times \mathbf{B}, \mathbf{C}\big]_v = \mathbf{A} \times \big[\mathbf{B}, \mathbf{C}\big]_v + \big[\mathbf{A}, \mathbf{C}\big]_v \times \mathbf{B}. \tag{16}$$

(c) Jacobi's identity:

$$\big[\mathbf{A}, \big[\mathbf{B}, \mathbf{C}\big]_v\big]_v + \big[\mathbf{B}, \big[\mathbf{C}, \mathbf{A}\big]_v\big]_v + \big[\mathbf{C}, \big[\mathbf{A}, \mathbf{B}\big]_v\big]_v = 0. \tag{17}$$

4. (**50 points.**) (Refer Dirac, Sec. 21.)
The product rule for Poisson braket can be stated in the following different forms:

$$\big[A_1 A_2, B\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} = A_1\big[A_2, B\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} + \big[A_1, B\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} A_2, \tag{18a}$$

$$\big[A, B_1 B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} = B_1\big[A, B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} + \big[A, B_1\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} B_2. \tag{18b}$$

Thus, evaluate, in two different ways,

$$\big[A_1 A_2, B_1 B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} = A_1 B_1\big[A_2, B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} + A_1\big[A_2, B_1\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} B_2 + B_1\big[A_1, B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} A_2 + \big[A_1, B_1\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} B_2 A_2, \quad (19$$

$$\big[A_1 A_2, B_1 B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} = B_1 A_1\big[A_2, B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} + B_1\big[A_1, B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} A_2 + A_1\big[A_2, B_1\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} B_2 + \big[A_1, B_1\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} A_2 B_2. \quad (19$$

Subtracting these results, obtain

$$(A_1 B_1 - B_1 A_1)\big[A_2, B_2\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} = \big[A_1, B_1\big]_{\mathbf{x},\mathbf{p}}^{\text{P.B.}} (A_2 B_2 - B_2 A_2), \tag{20}$$

Using the definition of the commutation relations,

$$\big[A, B\big] \equiv AB - BA, \tag{21}$$

thus obtain the relation

$$\big[A_1, B_1\big]\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big[A_2, B_2\big]. \tag{22}$$

Since this condition holds for the operators A_1 and B_1, independent of the operators A_2 and B_2, conclude that

$$\big[A_1, B_1\big] = i\hbar\big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{23}$$

$$\big[A_2, B_2\big] = i\hbar\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{24}$$

where $\hbar$ has to be a constant, independent of A_1, A_2, B_1, and B_2. This is the connection between the commutator braket in quantum mechanics and the Poisson braket in classical mechanics. The imaginary number $i = \sqrt{-1}$ is necessary because the construction

$$C = \frac{1}{i}(AB - BA) \tag{25}$$

is, by construction, Hermitian.

5. (**20 points.**) Given F and G are constants of motion, that is,

$$\big[F, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0 \qquad \text{and} \qquad \big[G, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{26}$$

Then, using Jacobi's identity, show that $\big[F, G\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}$ is also a constant of motion. Thus, conclude the following:

   (a) If L_x and L_y are constants of motion, then L_z is also a constant of motion.

   (b) If p_x and L_z are constants of motion, then p_y is also a constant of motion.

6. (**30 points.**) Hamiltonian for a charge particle in a uniform magnetic field is

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2, \qquad \mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}. \tag{27}$$

   (a) For a constant (homogenous in space) magnetic field $\mathbf{H}$, verify that

$$\mathbf{A} = \frac{1}{2}\mathbf{H} \times \mathbf{r} \tag{28}$$

   is a possible vector potential. That is, show that

$$\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}. \tag{29}$$

   (b) Evaluate the Hamilton equations of motion.

(c) Show that

$$\left[\mathbf{v}^i, \mathbf{v}^j\right]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{q}{m^2 c}\mathbf{1} \times \mathbf{B} \equiv \frac{q}{m^2 c}\varepsilon^{ijm} B_m, \tag{30}$$

which is sometimes expressed in the form

$$\mathbf{v} \times \mathbf{v} = \frac{q}{m^2 c}\mathbf{B} \tag{31}$$

using the fact that the vector product also satisfies the same Lie algebra.