Homework No. 04 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2016 Mar 1, 4.30pm

1. (**20 points.**) (Ref: Milton's notes.) Consider an atom entering a Stern-Gerlach apparatus. Deflection upward begins as soon as the atom enters the inhomogeneous field. By the time the atom leaves the field, it has been deflected upward by a net amount Δz. Compute Δz for

$$\mu_z = 10^{-27}\frac{\mathrm{J}}{\mathrm{G}}, \quad \frac{\partial H_z}{\partial z} = 10^6\frac{\mathrm{G}}{\mathrm{m}}, \quad l = 10\,\mathrm{cm}, \quad T = \frac{mv_x^2}{k} = 10^3\,\mathrm{K}. \tag{1}$$

2. (**20 points.**) The wavefunctions for the Stern-Gerlach experiment are

$$\psi_+(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix} \quad \text{and} \quad \psi_-(\theta,\phi) = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}. \tag{2}$$

   (a) Show that $\psi_-(\pi-\theta,\phi+\pi) = \psi_+(\theta,\phi)$, up to a phase.

   (b) What is the physical interpretation?

3. (**20 points.**) The probability for a measurement in the Stern-Gerlach experiment is given by

$$p\big([+;\theta_1,\phi_1] \to [\pm;\theta_2,\phi_2]\big) = \frac{1\pm\cos\Theta}{2}, \tag{3}$$

where

$$\cos\Theta = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2). \tag{4}$$

Verify that

$$\mathrm{tr}\left(\frac{1+\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}(\theta_1,\phi_1)}{2}\right)\left(\frac{1\pm\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}(\theta_2,\phi_2)}{2}\right) = \frac{1\pm\cos\Theta}{2}. \tag{5}$$

4. (**30 points.**) If $\boldsymbol{\sigma}$ is the vector constructed out of Pauli matrices and $\mathbf{a}$ is a (numerical) vector, evaluate

   (a)

$$\mathrm{tr}\cos(\boldsymbol{\sigma}\cdot\mathbf{a}). \tag{6}$$

   (b)

$$\mathrm{tr}\sin(\boldsymbol{\sigma}\cdot\mathbf{a}). \tag{7}$$

   (c)

$$\mathrm{tr}\tan(\boldsymbol{\sigma}\cdot\mathbf{a}). \tag{8}$$

Hint: $\mathrm{tr} f(A) = \sum_i f(a_i)$.

5. (**20 points.**) Evaluate

$$[(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b})]\,(\boldsymbol{\sigma}\cdot\mathbf{c}). \tag{9}$$

Then evaluate

$$(\boldsymbol{\sigma}\cdot\mathbf{a})\,[(\boldsymbol{\sigma}\cdot\mathbf{b})(\boldsymbol{\sigma}\cdot\mathbf{c})]\,. \tag{10}$$

Are they equal?

6. (**20 points.**) The Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

is written in the eigenbasis of σ_z. Write σ_x in the eigenbasis of σ_y.

7. (**30 points.**) The Pauli matrices satisfy

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \tag{12}$$

A particular representation of Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

In particular, these are Pauli matrices in the eigenbasis of σ_z.

(a) Construct the matrix

$$\sigma_{\theta,\phi} = \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}(\theta,\phi) = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}, \tag{14}$$

where

$$\boldsymbol{\sigma} = \sigma_x\hat{\mathbf{i}} + \sigma_y\hat{\mathbf{j}} + \sigma_z\hat{\mathbf{k}}, \tag{15}$$

$$\hat{\mathbf{n}}(\theta,\phi) = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}. \tag{16}$$

Find the eigenvalues $\sigma'_{\theta,\phi}$ and the normalized eigenvectors, $|\sigma'_{\theta,\phi} = +1\rangle$ and $|\sigma'_{\theta,\phi} = -1\rangle$, (up to a phase) of the matrix $\sigma_{\theta,\phi}$.

(b) Now compute the matrices

$$\bar{\sigma}_x = \begin{pmatrix} \langle\sigma'_{\theta,\phi} = +1|\sigma_x|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = +1|\sigma_x|\sigma'_{\theta,\phi} = -1\rangle \\ \langle\sigma'_{\theta,\phi} = -1|\sigma_x|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = -1|\sigma_x|\sigma'_{\theta,\phi} = -1\rangle \end{pmatrix}, \tag{17}$$

$$\bar{\sigma}_y = \begin{pmatrix} \langle\sigma'_{\theta,\phi} = +1|\sigma_y|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = +1|\sigma_y|\sigma'_{\theta,\phi} = -1\rangle \\ \langle\sigma'_{\theta,\phi} = -1|\sigma_y|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = -1|\sigma_y|\sigma'_{\theta,\phi} = -1\rangle \end{pmatrix}, \tag{18}$$

$$\bar{\sigma}_z = \begin{pmatrix} \langle\sigma'_{\theta,\phi} = +1|\sigma_z|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = +1|\sigma_z|\sigma'_{\theta,\phi} = -1\rangle \\ \langle\sigma'_{\theta,\phi} = -1|\sigma_z|\sigma'_{\theta,\phi} = +1\rangle & \langle\sigma'_{\theta,\phi} = -1|\sigma_z|\sigma'_{\theta,\phi} = -1\rangle \end{pmatrix}. \tag{19}$$

These matrices are representation of the Pauli matrices in the eigenbasis of $\sigma_{\theta,\phi}$.

(c) Show that

$$\bar{\sigma}_i\bar{\sigma}_j = \delta_{ij} + i\varepsilon_{ijk}\bar{\sigma}_k. \tag{20}$$