Homework No. 06 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2016 Mar 29, 4.30pm

1. (**40 points.**) Robertson's generalization of Heisenberg's uncertainty relation

$$(\delta A)(\delta B) \geq \frac{1}{2}|\langle C \rangle|, \tag{1}$$

for $(A, B) = (\sigma_x, \sigma_y)$ reads

$$(\delta \sigma_x)(\delta \sigma_y) \geq |\langle \sigma_z \rangle|. \tag{2}$$

   (a) Show that minimal uncertainty states $|\text{min}\rangle$, characterized by the equality

$$(\delta \sigma_x)(\delta \sigma_y) = |\langle \sigma_z \rangle|, \tag{3}$$

   satisfies matrix equations

$$(\sigma_x \pm i\sigma_y)|\text{min}\rangle = 0. \tag{4}$$

   Hint: Note that $\sigma_x\sigma_y + \sigma_y\sigma_x = 0$.

   (b) Determine the two (normalized) minimum uncertainty states $|\text{min}\rangle$. Are linear combinations of the two states minimal uncertainty states?

   (c) Evaluate $\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$, $\langle \sigma_z \rangle$, $\langle \sigma_x^2 \rangle$, $\langle \sigma_y^2 \rangle$, $\langle \sigma_z^2 \rangle$, $\delta\sigma_x$, $\delta\sigma_y$, and $\delta\sigma_z$, when the system is in the minimum uncertainty state.

   (d) Verify Eq. (3).

2. (**20 points.**) The minimum uncertainty state for Heisenberg's uncertainty relation

$$\delta q \delta p \geq \frac{1}{2} \tag{5}$$

in the position eigenbasis is, for $\langle q \rangle = \langle p \rangle = 0$ and $\delta q = \delta p = 1/\sqrt{2}$,

$$\psi_0(q') = \frac{1}{\pi^{\frac{1}{4}}} e^{-\frac{1}{2}q'^2}. \tag{6}$$

Evaluate the minimum uncertainty state in the momentum eigenbasis, $\psi_0(p')$, by evaluating the integral for the Fourier transform

$$\psi_0(p') = \int_{-\infty}^{\infty} \frac{dq'}{\sqrt{2\pi}} e^{-iq'p'} \psi_0(q'). \tag{7}$$

3. (**30 points.**) Show that

$$\delta(q' - \langle q \rangle) = \lim_{\delta q \to 0} \frac{1}{\sqrt{\pi}} \frac{1}{2\delta q} e^{-\left(\frac{q' - \langle q \rangle}{2\delta q}\right)^2} \tag{8}$$

is a suitable representation for the Dirac δ-function. That is, verify that it satisfies

$$\delta(q' - \langle q \rangle) \to 0, \qquad \text{for} \qquad q' \neq \langle q \rangle, \tag{9a}$$
$$\delta(q' - \langle q \rangle) \to \infty, \qquad \text{for} \qquad q' = \langle q \rangle, \tag{9b}$$

and

$$\int_{-\infty}^{\infty} dq' \delta(q' - \langle q \rangle) = 1. \tag{10}$$