Homework No. 07 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2016 Apr 5, 4.30pm

1. (**50 points.**) A quantum harmonic oscillator can be constructed out of two non-Hermitian operators, y and $y^\dagger$, that satisfy the commutation relation

$$[y, y^\dagger] = 1. \tag{1}$$

The eigenstate spectrum of the (Hermitian) number operator, $N = y^\dagger y$, represented by $|n\rangle$, where $n = 0, 1, 2, \ldots$, satisfy

$$N|n\rangle = n|n\rangle, \qquad y|n\rangle = \sqrt{n}|n-1\rangle, \qquad y^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2}$$

(a) Build the matrix representation of the lowering operator using

$$\langle n|y|n'\rangle = \begin{bmatrix} \langle 0|y|0\rangle & \langle 0|y|1\rangle & \langle 0|y|2\rangle & \langle 0|y|3\rangle & \langle 0|y|4\rangle & \cdots \\ \langle 1|y|0\rangle & \langle 1|y|1\rangle & \langle 1|y|2\rangle & \langle 1|y|3\rangle & \langle 1|y|4\rangle & \cdots \\ \langle 2|y|0\rangle & \langle 2|y|1\rangle & \langle 2|y|2\rangle & \langle 2|y|3\rangle & \langle 2|y|4\rangle & \cdots \\ \langle 3|y|0\rangle & \langle 3|y|1\rangle & \langle 3|y|2\rangle & \langle 3|y|3\rangle & \langle 3|y|4\rangle & \cdots \\ \langle 4|y|0\rangle & \langle 4|y|1\rangle & \langle 4|y|2\rangle & \langle 4|y|3\rangle & \langle 4|y|4\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3}$$

Kindly calculate the first 5×5 block of the infinite dimensional matrix to report the pattern in the following questions.

(b) Similarly, build the matrix representation of the raising operator $y^\dagger$.

(c) Build the matrix representation of the number operator N.

(d) Using the constructions

$$y = \frac{1}{\sqrt{2\hbar}}(x + ip) \quad \text{and} \quad y^\dagger = \frac{1}{\sqrt{2\hbar}}(x - ip), \tag{4}$$

determine the matrix representations for the Hermitian operators, x and p. Check that x and p are Hermitian matrices.

(e) Determine the matrices for the operators xp and px, and verify the commutation relation

$$\frac{1}{i\hbar}[x, p] = 1. \tag{5}$$

2. (**10 points.**) Using the asymptotic form for Hermite polynomials for large n,

$$e^{-\frac{x^2}{2}} H_n(x) \xrightarrow{n \gg 1} \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \cos\left(x\sqrt{2n} - n\frac{\pi}{2}\right), \tag{6}$$

discuss the manner in which the harmonic oscillator eigenfunctions

$$\psi_n(x) = \frac{1}{\sqrt{2^n \hbar^n \sqrt{\pi} n!}} e^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) \tag{7}$$

approach those of the free particle, satisfying the differential equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi_n(x) = E_n \phi_n(x) \tag{8}$$

with eigenfunctions

$$\phi_n(x) = A \cos\left(\sqrt{\frac{2mE_n}{\hbar^2}} x + \delta\right), \tag{9}$$

in the limit when the frequency of oscillations $\omega \to 0$.

3. (**40 points.**) (Set $\hbar = 1$.) From

$$y|n\rangle = \sqrt{n}|n-1\rangle \tag{10}$$

derive

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x). \tag{11}$$

Check this for $n = 4, 3, 2, 1, 0$. From

$$y^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle \tag{12}$$

derive

$$\left(2x - \frac{d}{dx}\right) H_n(x) = H_{n+1}(x). \tag{13}$$

Add the two statements to obtain

$$2x H_n(x) = H_{n+1}(x) + 2n H_{n-1}(x). \tag{14}$$

This recursion relation gives a way of recursively calculating $H_{n+1}(x)$ in terms of $H_n(x)$ and $H_{n-1}(x)$. Check this for $n = 3, 2, 1, 0$.

4. (**20 points.**) Use the results of Problem 3 to deduce the differential equation

$$\left(\frac{d^2}{dx^2} - 2x\frac{d}{dx} + 2n\right) H_n(x) = 0. \tag{15}$$

Show the equivalence of this with

$$\left(\frac{d^2}{dx^2} - x^2 + 2n + 1\right) \psi_n(x) = 0. \tag{16}$$

This is the "time-independent Schrödinger equation" for the harmonic oscillator.