Homework No. 08 (Spring 2016)

PHYS 530A: Quantum Mechanics II

Due date: Tuesday, 2016 Apr 19, 4.30pm

1. (**20 points.**) A vector operator $\mathbf{V}$ is defined by the transformation property

$$\frac{1}{i\hbar}\left[\mathbf{V}, \delta\boldsymbol{\omega}\cdot\mathbf{J}\right] = \delta\boldsymbol{\omega}\times\mathbf{V}, \tag{1}$$

which states the commutation relations of components of $\mathbf{V}$ with those of angular momentum $\mathbf{J}$. Since a scalar operator S does not change under rotations it is defined by the corresponding transformation

$$\frac{1}{i\hbar}\left[S, \delta\boldsymbol{\omega}\cdot\mathbf{J}\right] = 0. \tag{2}$$

   (a) Using Eq. (1), show that the scalar product of two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a scalar. That is,

$$\frac{1}{i\hbar}\left[\mathbf{V}_1\cdot\mathbf{V}_2, \delta\boldsymbol{\omega}\cdot\mathbf{J}\right] = 0. \tag{3}$$

   (b) Using Eq. (1), show that the vector product of vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is a vector. That is,

$$\frac{1}{i\hbar}\left[\mathbf{V}_1\times\mathbf{V}_2, \delta\boldsymbol{\omega}\cdot\mathbf{J}\right] = \delta\boldsymbol{\omega}\times(\mathbf{V}_1\times\mathbf{V}_2). \tag{4}$$

2. (**50 points.**) For $j = 1$:

   (a) Determine the matrix representantion for

$$J_z, J_x, J_y, J_+, J_-, \text{and } J^2. \tag{5}$$

   For example,

$$J_z = \begin{bmatrix} \langle j,j|J_z|j,j\rangle & \langle j,j|J_z|j,j-1\rangle & \cdots & \langle j,j|J_z|j,-j\rangle \\ \langle j,j-1|J_z|j,j\rangle & \langle j,j-1|J_z|j,j-1\rangle & \cdots & \langle j,j-1|J_z|j,-j\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle j,-j|J_z|j,j\rangle & \langle j,-j|J_z|j,j-1\rangle & \cdots & \langle j,-j|J_z|j,-j\rangle \end{bmatrix}. \tag{6}$$

   (b) Evaluate

$$\text{Tr}(J_k), \quad \text{Tr}(J_kJ_l), \quad \text{and} \quad \text{Tr}(J_k^2J_l^2), \quad \text{for} \quad k,l = x,y,z. \tag{7}$$

3. (**30 points.**) Consider the construction

$$K(\lambda) = e^{-\lambda A} B e^{\lambda A} \tag{8}$$

in terms of two operators A and B. Show that

$$\frac{\partial K}{\partial \lambda} = [K, A]. \tag{9}$$

Evaluate the higher derivatives

$$\frac{\partial^n K}{\partial \lambda^n} \tag{10}$$

recursively. Thus, using Taylor expansion around $\lambda = 0$, show that

$$K(\lambda) = B + \lambda [B, A] + \frac{\lambda^2}{2!}[[B, A], A] + \frac{\lambda^3}{3!}[[[B, A], A], A] + \ldots. \tag{11}$$

4. (**40 points.**) Consider the following unitary transformations,

$$J_x(\phi) = e^{-\frac{i}{\hbar}\phi J_z} J_x e^{\frac{i}{\hbar}\phi J_z} \tag{12}$$

and

$$J_y(\phi) = e^{-\frac{i}{\hbar}\phi J_z} J_y e^{\frac{i}{\hbar}\phi J_z}. \tag{13}$$

By differentiating with respect to ϕ, and solving the resulting differential equations, derive

$$J_x(\phi) = J_x \cos\phi + J_y \sin\phi, \tag{14a}$$

$$J_y(\phi) = -J_x \sin\phi + J_y \cos\phi. \tag{14b}$$

Further, derive

$$J_+(\phi) = e^{-i\phi} J_+ \quad \text{and} \quad J_-(\phi) = e^{i\phi} J_-. \tag{15}$$