Final Exam (Fall 2016)

PHYS 530B: Quantum Mechanics II

Date: 2016 Dec 14

1. The number density of electrons in a thin film of thickness d,

$$n_f = \frac{n_{\text{tot}}}{L_x L_y d}, \tag{1}$$

as a function of the Fermi energy E_F is conveniently expressed in terms of the parameter

$$N = \sqrt{\frac{2m^* E_F d^2}{\hbar^2 \pi^2}} = \frac{k_F d}{\pi} = \frac{2d}{\lambda_F}, \tag{2}$$

where k_F is the Fermi wave-vector and λ_F is the Fermi wavelength, by the expression

$$n_f(N) = n_f(\infty)\nu(x), \tag{3}$$

where

$$n_f(\infty) = \frac{\pi}{2d^3}\frac{2}{3}N^3 = \frac{k_F^3}{3\pi^2} \tag{4}$$

and

$$\nu(x) = \frac{3}{2}\left(x - \frac{1}{3}x^3\right) + \frac{3}{2N}\left(1 - \frac{1}{2}x^2\right) - \frac{1}{4N^2}x, \qquad x = \frac{[N]}{N}. \tag{5}$$

Here $[N]$ is the integer part of N.

(a) Plot $x = [N]/N$ as a function of N, for $0 < N < 10$. In particular, consider regions $0 \le x < 1.2$.

(b) Plot $\nu(x)$ as a function of N, for $0 < N < 10$. In particular, consider regions $0.8 \le \nu(x) < 2.0$.

(c) Plot $\nu(x)$ as a function of $1/N$, for $0 < 1/N < 0.5$. In particular, consider regions $0.9 \le \nu(x) < 1.5$.

2. The electronic structure of an isolated carbon atom is $1s^2 2s^2 2p^2$. In graphene the carbon atoms arrange themselves in a single two-dimensional hexagonal pattern. The carbon atoms are 0.142 nm apart. Each carbon atom in graphene shares three electrons with three closest neighbours in sp^2 hydrid orbitals. The excitations of the remaining electron, corresponding to the remaining p-orbital oriented out of the plane (π-bond), is described by the Hamiltonian

$$H = v_F \boldsymbol{\alpha} \cdot \mathbf{p}, \tag{6}$$

where $\mathbf{p}$ represents the momentum in two dimensions ($p_z = 0$) and $v_F/c \sim 300$. In terms of the rescaled momentum

$$\tilde{p}^\mu = \left(\frac{H}{c}, \frac{v_F}{c}\mathbf{p}\right), \qquad \mu = 0, 1, 2, \tag{7}$$

and Dirac's Gamma matrices $\gamma^\mu = (\beta, \beta\alpha)$, for $\mu = 0, 1, 2$, we have the (massless) Dirac equation

$$\gamma^\mu \tilde{p}_\mu = 0. \tag{8}$$

The dielectric permittivity in this (quantum field theoretical) model is given in terms of vacuum polarization effects using the relation

$$\Pi^{ij}(0, \omega) = \delta^{ij}\omega^2(\varepsilon - 1), \tag{9}$$

where the vacuum polarization tensor is given by

$$i\Pi^{\mu\nu}(\tilde{\mathbf{k}}, \omega) = -4\pi\alpha \,\mathrm{tr} \int \frac{d^3\tilde{p}}{(2\pi)^3}\, \gamma^\mu \frac{1}{\gamma \cdot \tilde{p}} \gamma^\nu \frac{1}{\gamma \cdot (\tilde{p} - \tilde{k})}, \tag{10}$$

$\alpha = e^2/4\pi\hbar c$ is the fine structure constant.

(a) Using the idea of rationalizing denominators show that

$$\frac{1}{\gamma \cdot \tilde{p}} = -\frac{\gamma \cdot \tilde{p}}{\tilde{p}^2}. \tag{11}$$

(b) Use Eq. (11) to move all the Gamma matrices to the numerator in the expression for vacuum polarization tensor in Eq. (10). Evaluate the trace

$$\mathrm{tr}(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta) = 4(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}) \tag{12}$$

and then show that

$$i\Pi^{\mu\nu}(\tilde{\mathbf{k}}, \omega) = -16\pi\alpha \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{\left[\tilde{p}^\mu(\tilde{p} - \tilde{k})^\nu + \tilde{p}^\nu(\tilde{p} - \tilde{k})^\mu - g^{\mu\nu}\tilde{p} \cdot (\tilde{p} - \tilde{k})\right]}{\tilde{p}^2(\tilde{p} - \tilde{k})^2}, \tag{13}$$

(c) Show that the inverse of an operator has the integral representation

$$\frac{1}{M} = \lim_{\delta \to 0} \frac{1}{M - i\delta} = \lim_{\delta \to 0} i \int_0^\infty ds\, e^{-is(M - i\delta)}, \tag{14}$$

where the parameter δ is a positive real number. Thus, show that

$$\frac{1}{\tilde{p}^2(\tilde{p} - \tilde{k})^2} = \lim_{\delta \to 0} \int_0^\infty ds_1 e^{-is_1(\tilde{p}^2 - i\delta)} \int_0^\infty ds_2 e^{-is_2[(\tilde{p} - \tilde{k})^2 - i\delta]}. \tag{15}$$

Using substitutions

$$s_1 = s(1-u), \tag{16a}$$
$$s_2 = su, \tag{16b}$$

deduce $ds_1 ds_2 = sdsdu$, where $0 \leq s < \infty$, $0 \leq u \leq 1$, and show that

$$\frac{1}{\tilde{p}^2(\tilde{p}-\tilde{k})^2} = \lim_{\delta\to 0}\int_0^\infty sds\, e^{-s\delta}\int_0^1 du\, e^{-is(1-u)\tilde{p}^2} e^{-isu(\tilde{p}-\tilde{k})^2} \tag{17a}$$
$$= \lim_{\delta\to 0}\int_0^\infty sds\, e^{-s\delta}\int_0^1 du\, e^{-isu(1-u)\tilde{k}^2} e^{-is(\tilde{p}-u\tilde{k})^2}. \tag{17b}$$

Thus, show that

$$i\Pi^{\mu\nu}(\tilde{\mathbf{k}},\omega) = \frac{2\alpha}{\pi^2}\lim_{\delta\to 0}\int_0^\infty sds\, e^{-s\delta}\int_0^1 du\, e^{-isu(1-u)\tilde{k}^2} \times \int d^3\tilde{p}\left[\tilde{p}^\mu(\tilde{p}-\tilde{k})^\nu + \tilde{p}^\nu(\tilde{p}-\tilde{k})^\mu - g^{\mu\nu}\tilde{p}\cdot(\tilde{p}-\tilde{k})\right] e^{-is(\tilde{p}-u\tilde{k})^2}. \tag{18}$$

(d) Show that, replacing $\tilde{p} - u\tilde{k} \to \tilde{p}$ leads to the integral

$$\int d^3\tilde{p}\left[(\tilde{p}+u\tilde{k})^\mu(\tilde{p}-(1-u)\tilde{k})^\nu + (\tilde{p}+u\tilde{k})^\nu(\tilde{p}-(1-u)\tilde{k})^\mu - g^{\mu\nu}(\tilde{p}+u\tilde{k})\cdot(\tilde{p}-(1-u)\tilde{k})\right] e^{-is\tilde{p}^2}. \tag{19}$$

Evaluate this integral by computing the following integrals, (in terms of the Gaussian integral,)

$$\int d^3\tilde{p}\, e^{-is\tilde{p}^2} = \frac{1}{i}\left(\frac{\pi}{is}\right)^{\frac{3}{2}}, \tag{20a}$$
$$\int d^3\tilde{p}\left[\tilde{p}^\mu \tilde{k}^\nu\right] e^{-is\tilde{p}^2} = 0, \tag{20b}$$
$$\int d^3\tilde{p}\left[\tilde{p}^\mu \tilde{p}^\nu\right] e^{-is\tilde{p}^2} = -g^{\mu\nu}\frac{1}{2s}\left(\frac{\pi}{is}\right)^{\frac{3}{2}}, \tag{20c}$$
$$\int d^3\tilde{p}\left[\tilde{p}^2\right] e^{-is\tilde{p}^2} = -\frac{3}{2s}\left(\frac{\pi}{is}\right)^{\frac{3}{2}}. \tag{20d}$$

Thus, show that

$$i\Pi^{\mu\nu}(\tilde{\mathbf{k}},\omega) = (2\tilde{k}^\mu\tilde{k}^\nu - g^{\mu\nu}\tilde{k}^2)\frac{2\alpha}{\sqrt{i\pi}}\lim_{\delta\to 0}\int_0^1 du\, u(1-u)\int_0^\infty \frac{ds}{s}\sqrt{s}\, e^{-s[\delta + iu(1-u)\tilde{k}^2]} + g^{\mu\nu}\frac{\alpha}{i\sqrt{i\pi}}\lim_{\delta\to 0}\int_0^1 du\int_0^\infty \frac{ds}{s}\frac{1}{\sqrt{s}}\, e^{-s[\delta + iu(1-u)\tilde{k}^2]} \tag{21}$$

(e) Evaluate the s-integrals using the integral representation for the gamma function

$$\Gamma(z) = \int_0^\infty ds s^{z-1} e^{-s}, \tag{22}$$

and evaluate the following u-integrals using the integral representation for the beta function

$$\mathrm{B}(x,y) = \int_0^1 du\, u^{x-1}(1-u)^{y-1}. \tag{23}$$

In particular, $-\frac{1}{2}\Gamma(-\frac{1}{2}) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\mathrm{B}(\frac{3}{2},\frac{3}{2}) = \pi/8$. Thus, evaluate the vacuum polarization tensor

$$\Pi^{\mu\nu}(\tilde{\mathbf{k}},\omega) = -\frac{\pi\alpha}{2\tilde{k}}(\tilde{k}^\mu \tilde{k}^\nu - g^{\mu\nu}\tilde{k}^2). \tag{24}$$

(f) Show that, using $\tilde{k}^2 = v_F^2 \mathbf{k}\cdot\mathbf{k} - \omega^2$,

$$\Pi^{00}(0,\omega) = 0, \tag{25a}$$

$$\Pi^{ij}(0,\omega) = -\frac{\pi\alpha}{2i\omega}\delta^{ij}\omega^2. \tag{25b}$$

Read out the dielectric permittivity of graphene, using Eq. (9), as

$$\varepsilon - 1 = -2 \times \frac{\pi\alpha}{2i\omega} = -\frac{\pi\alpha}{i\omega}, \tag{26}$$

where we multiplied by a factor of 2 for the two independent excitations (representing the Dirac points K and K').