Homework No. 02 (Fall 2016)

PHYS 530B: Quantum Mechanics II

Due date: Tuesday, 2016 Sep 13, 4.30pm

1. (**50 points.**) (Let $\hbar = 0$. That is, we are discussing a classical hydrogenic atom.) The Hamiltonian for a hydrogenic atom is

$$H(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2\mu} - \frac{Ze^2}{r}. \tag{1}$$

The Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the axial vector $\mathbf{A}$, are conserved quantities for a hydrogenic atom.

(a) Show that

$$\mathbf{W} = \frac{\mu Z e^2}{L^2} \mathbf{A} \times \mathbf{L} \tag{2}$$

is also a conserved quantity. That is, show that $d\mathbf{W}/dt = 0$. Thus, together, the vectors $\mathbf{L}$, $\mathbf{A}$, and $\mathbf{W}$, form an orthogonal set that remain fixed in time. Show that the vector $\mathbf{W}$ can be expressed in the form

$$\mathbf{W} = \mathbf{p} + \frac{\mu Z e^2}{L^2} \hat{\mathbf{r}} \times \mathbf{L}. \tag{3}$$

Further, show that

$$W = \mu Z e^2 \frac{A}{L}. \tag{4}$$

(b) Determine the components of the momentum $\mathbf{p}$ along these orthogonal vectors by evaluating $(\mathbf{p} \cdot \hat{\mathbf{L}})$, $(\mathbf{p} \cdot \hat{\mathbf{A}})$, and $(\mathbf{p} \cdot \hat{\mathbf{W}})$. Thus, construct the momentum $\mathbf{p}$ in the form

$$\mathbf{p} = (\mathbf{p} \cdot \hat{\mathbf{L}})\, \hat{\mathbf{L}} + (\mathbf{p} \cdot \hat{\mathbf{A}})\, \hat{\mathbf{A}} + (\mathbf{p} \cdot \hat{\mathbf{W}})\, \hat{\mathbf{W}}. \tag{5}$$

(c) It is well known that the position $\mathbf{r}$ traverses an ellipse about the origin. This is the content of Kepler's first law of motion. Show that the momentum $\mathbf{p}$ traverses a circle about a fixed point $\mathbf{p}_0$. That is, show that the momentum $\mathbf{p}$ satisfies the equation of a circle,

$$|\mathbf{p} - \mathbf{p}_0| = q. \tag{6}$$

(d) Determine the vector $\mathbf{p}_0$ representing the center of this circle, and find the radius q of this circle. Verify that the center $\mathbf{p}_0$ is a conserved quantity.