Homework No. 04 (Fall 2016)

PHYS 530B: Quantum Mechanics II

Due date: Thursday, 2016 Oct 20, 4.30pm

1. (**30 points.**) Find the two roots of the quadratic equation

$$\varepsilon x^2 - x + 1 = 0. \tag{1}$$

Show that the series expansion of the roots in the variable ε are

$$x = \begin{cases} 1 + \varepsilon + \mathcal{O}(\varepsilon)^2, \\ \dfrac{1}{\varepsilon} - 1 + \mathcal{O}(\varepsilon). \end{cases} \tag{2}$$

Let us treat the quadratic equation in Eq. (1) as a perturbation to the trivial equation

$$x_0 = 1, \tag{3}$$

obtained by setting $\varepsilon = 0$ in Eq. (1). To find the roots to the leading order in the perturbation parameter ε we use the ansatz

$$x = x_0 + a_1\varepsilon + \mathcal{O}(\varepsilon)^2. \tag{4}$$

Substitute Eq. (4) in Eq. (1) to show that $a_1 = 1$. Thus, we learn that

$$x = 1 + \varepsilon + \mathcal{O}(\varepsilon)^2, \tag{5}$$

which is indeed one of the root, to the leading order in ε. What about the other root? To this end, let us make the change of variables using the substitution

$$x = \frac{y}{\varepsilon}. \tag{6}$$

Show that this leads to the quadratic equation

$$y^2 - y + \varepsilon = 0. \tag{7}$$

The unperturbed equation, obtained by setting $\varepsilon = 0$, is

$$y_0^2 - y_0 = 0, \tag{8}$$

whose roots are $y_0 = 0, 1$. Use the ansatz

$$y = y_0 + b_1\varepsilon + b_2\varepsilon^2 + \mathcal{O}(\varepsilon)^3 \tag{9}$$

to derive

$$b_1 = \frac{1}{1-2y_0}, \quad b_2 = \frac{b_1^2}{1-2y_0}. \tag{10}$$

Thus, reproduce the results in Eq. (2), this time using the techniques of perturbation theory. Using the ideas described above find the five roots of the quintic equation

$$\varepsilon x^5 - x + 1 = 0, \tag{11}$$

to the leading order in ε. Verify your solutions for $\varepsilon = 0.001$, say using Mathematica. Abel's impossibility theorem rules out algebraic solutions to general quintic equations.

2. (**20 points.**) Consider the eigenvalue problem for a Hermitian matrix A,

$$A|A'\rangle = A'|A'\rangle, \tag{12}$$

where $|A'\rangle$ form an orthonormal set of eigenfunctions. Variation, with respect to an arbitrary parameter, δA in the operator leads to the relation

$$\delta A|A'\rangle + A\delta|A'\rangle = \delta A'|A'\rangle + A'\delta|A'\rangle. \tag{13}$$

(a) Thus, derive the variation in the eigenvalue $\delta A'$ to be given by

$$\delta A' = \langle A'|\delta A|A'\rangle, \tag{14}$$

(b) and the variation in the eigenfunction $\delta|A'\rangle$, given in terms of its projection on the original eigenfunctions, to be

$$\langle A''|\delta\left(|A'\rangle\right) = \frac{\langle A''|\delta A|A'\rangle}{A' - A''}, \qquad A' \neq A''. \tag{15}$$

In particular, we can express the variation in the eigenfunction in terms of the original eigenfunctions as

$$\delta|A'\rangle = \sum_{A''} |A''\rangle\langle A''|\left(\delta|A'\rangle\right). \tag{16}$$

Further, since

$$\delta\langle A'|A'\rangle = 0, \tag{17}$$

one chooses

$$\langle A'|\delta\left(|A'\rangle\right) = 0. \tag{18}$$

3. (**20 points.**) Show that the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{19}$$

has eigenvalues $A_1' = e^{+i\theta}$ and $A_2' = e^{-i\theta}$, and the corresponding eigenfunctions

$$|A_1'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad |A_2'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{20}$$

Consider the perturbation

$$A + \delta A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} + \alpha \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}. \tag{21}$$

Determine the eigenvalues and eigenfunctions of $A + \delta A$ to the leading order in α.

4. (**20 points.**) Consider the unperturbed Hamiltonian to be

$$H = J_z, \tag{22}$$

where $\mathbf{J}$ is the angular momentum. Let $j = 1/2$. Consider the perturbation

$$H + \delta H = J_z + \alpha J_x. \tag{23}$$

Determine the eigenvalues and eigenfunctions of $H + \delta H$ to the leading order in α. Further, verify that the eigenfunctions are orthonormal upto the leading order in α.