Homework No. 05 (Fall 2016)

PHYS 530B: Quantum Mechanics II

Due date: Monday, 2016 Nov 7, 4.30pm

1. (**20 points.**) An arbitrary 2×2 matrix M with complex numbers as elements can be expressed as a linear combination of Pauli matrices $\boldsymbol{\sigma}$ and the identity matrix in the form

$$M = a_0 + \mathbf{a} \cdot \boldsymbol{\sigma}. \tag{1}$$

Show that the coefficients a_0 and $\mathbf{a}$ satisfy the relations

$$\mathrm{tr}(M) = 2a_0, \tag{2a}$$
$$\mathrm{tr}(M\boldsymbol{\sigma}) = 2\mathbf{a}. \tag{2b}$$

As an illustration, write

$$\sigma^i \sigma^j \sigma^k = A^{ijk} + B^{ijk}{}_m \sigma^m, \tag{3}$$

and determine the coefficients A^{ijk} and $B^{ijk}{}_m$.

2. (**20 points.**) Quaternions are extensions of complex numbers, like complex numbers are extensions of real numbers. A quaternion P can be expressed in terms of Pauli matrices as

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}. \tag{4}$$

(a) Show that the (Hamilton) product of two quaternions,

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}, \tag{5a}$$
$$Q = b_0 - i\mathbf{b} \cdot \boldsymbol{\sigma}, \tag{5b}$$

is given by

$$PQ = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) - i(a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{6}$$

(b) Verify that the Hamilton product is non-commutative. Determine $\big[P, Q\big]$.

3. (**120 points.**) Clifford-Dirac algebra:

(a) Let β and α^i, where $i = 1, 2, 3$, be four matrices that obey the algebra

$$\beta^2 = 1, \qquad \frac{1}{2}\big\{\alpha^i, \alpha^j\big\} = \delta^{ij}, \qquad \big\{\beta, \alpha^i\big\} = 0. \tag{7}$$

Deduce the trace identities

$$\mathrm{tr}(\beta) = 0, \tag{8a}$$
$$\mathrm{tr}(\alpha^i) = 0, \tag{8b}$$
$$\mathrm{tr}(\beta\alpha^i) = 0. \tag{8c}$$

(b) Define the Dirac matrices γ^μ, where $\mu = 0, 1, 2, 3$, using the definitions

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha^i. \tag{9}$$

i. Using Eq. (7), show that the algebra of the Dirac matrices is given by

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = -g^{\mu\nu} \tag{10}$$

ii. and the corresponding trace identities are contained in the relation

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = -4g^{\mu\nu}. \tag{11}$$

(c) Define the matrix

$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{12}$$

i. Deduce the following algebraic properties

$$\gamma_5^2 = -1, \qquad \{\gamma^\mu, \gamma_5\} = 0 \tag{13}$$

ii. and the corresponding trace identities

$$\mathrm{tr}(\gamma_5) = 0, \tag{14a}$$
$$\mathrm{tr}(\gamma_5\gamma^\mu) = 0, \tag{14b}$$
$$\mathrm{tr}(\gamma_5^2) = -4. \tag{14c}$$

iii. Further, verify the trace identity

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma_5) = 0. \tag{15}$$

(d) i. Show that

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha) = 0. \tag{16}$$

Hint: Introduce $\gamma_5^2 = -1$ inside the trace, and use the cyclic property of trace.

ii. Show that the trace of odd number of γ^μ matrices vanishes.

iii. Show that

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}). \tag{17}$$

(e) Define the matrices

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{18}$$

The components of $\sigma^{\mu\nu}$ comprise the six ways of multiplying together two different γ^μ matrices.

i. Verify that $\sigma^{\mu\nu}$ is antisymmetric under the interchange of indices μ and ν.

ii. Show that

$$\sigma^{0k} = i\alpha^k. \tag{19}$$

iii. Show that

$$\sigma^{ij} = -\frac{i}{2}\left[\alpha^i, \alpha^j\right] = \varepsilon^{ijk}\Sigma^k, \tag{20}$$

with the last equality serving as the definition of Σ^k.

iv. Combining the commutation and anti-commutation relations for γ^μ matrices, deduce the relation

$$\gamma^\mu\gamma^\nu = -g^{\mu\nu} - i\sigma^{\mu\nu}, \tag{21}$$

which is the generalization of the corresponding relation for Pauli matrices

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k. \tag{22}$$

v. Derive the trace identities

$$\mathrm{tr}(\sigma^{\mu\nu}) = 0, \tag{23a}$$
$$\mathrm{tr}(\sigma^{\mu\nu}\gamma^\alpha) = 0, \tag{23b}$$
$$\mathrm{tr}(\sigma^{\mu\nu}\gamma_5) = 0, \tag{23c}$$
$$\mathrm{tr}(\sigma^{\mu\nu}\sigma^{\alpha\beta}) = 4(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}). \tag{23d}$$

(f) Define the matrices

$$(i\gamma^\mu\gamma_5). \tag{24}$$

The components of $(i\gamma^\mu\gamma_5)$ comprise the four ways of multiplying together three different γ^μ matrices.

i. Derive the algebraic property

$$(i\gamma^\mu\gamma_5)(i\gamma^\nu\gamma_5) = -\gamma^\mu\gamma^\nu. \tag{25}$$

ii. Derive the trace identities

$$\mathrm{tr}(i\gamma^\mu\gamma_5) = 0, \tag{26a}$$
$$\mathrm{tr}((i\gamma^\mu\gamma_5)\gamma^\alpha) = 0, \tag{26b}$$
$$\mathrm{tr}((i\gamma^\mu\gamma_5)\gamma_5) = 0, \tag{26c}$$
$$\mathrm{tr}((i\gamma^\mu\gamma_5)\sigma^{\alpha\beta}) = 0, \tag{26d}$$
$$\mathrm{tr}((i\gamma^\mu\gamma_5)(i\gamma^\nu\gamma_5)) = 4g^{\mu\nu}. \tag{26e}$$

iii. Further, in the trace identity

$$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = c\,\varepsilon^{\mu\nu\alpha\beta} \tag{27}$$

find c.

(g) An arbitrary 4×4 matrix M with complex elements can be expressed in terms of the 16 independent matrices above

$$M = a + b_\mu\gamma^\mu + c\gamma_5 + d_\mu i\gamma^\mu\gamma_5 + \frac{1}{2}e_{\mu\nu}\sigma^{\mu\nu}. \tag{28}$$

Using the trace identities show the coefficients are given by

$$\mathrm{tr}(M) = 4a, \tag{29a}$$
$$\mathrm{tr}(M\gamma^\alpha) = -4b^\alpha, \tag{29b}$$
$$\mathrm{tr}(M\gamma_5) = -4c, \tag{29c}$$
$$\mathrm{tr}(Mi\gamma^\alpha\gamma_5) = 4d^\alpha, \tag{29d}$$
$$\mathrm{tr}(M\sigma^{\alpha\beta}) = 4e^{\alpha\beta}. \tag{29e}$$