Final Exam (Spring 2017)

PHYS 510: Classical Mechanics

Date: 2017 May 11

1. (**20 points.**) A catenary is the curve that an idealized hanging chain assumes under its own weight when supported only at its ends in a uniform gravitational field. It is the curve $y(x)$ that minimizes the potential energy of the hanging chain. Let us assume the two end points of the chain are at the same height. A catenary is given by

$$y = a \cosh \frac{x}{a}, \tag{1}$$

where the parameter a, an integration constant, characterizes the catenary.

   (a) Determine the perimeter length P of the hanging chain using

$$P = \int_{-x_0}^{x_0} ds. \tag{2}$$

   (b) Find the relation between the parameter a, the perimeter length P of the chain, and the height y_0 in Figure 1.

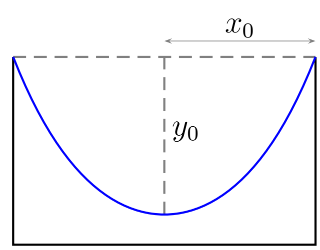

Figure 1: Problem 1.

2. (**40 points.**) Kepler problem is described by the potential energy

$$U(r) = -\frac{\alpha}{r}, \tag{3}$$

and the corresponding Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}. \tag{4}$$

For the case when the total energy E is negative,

$$-\frac{\alpha}{2r_0} < E < 0, \qquad r_0 = \frac{L_z^2}{\mu\alpha}, \tag{5}$$

where L_z is the angular momentum, the motion is described by an ellipse,

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \qquad e = \sqrt{1 + \frac{E}{(\alpha/2r_0)}}. \tag{6}$$

Perihelion is the point in the orbit of a planet at which it is closest to the Sun. The precession of the perihelion is suitably defined in terms of the displacement $\Delta\phi$ of the perihelion during one revolution,

$$\Delta\phi = -2\pi + 2\int_{r_{\min}}^{r_{\max}} d\phi, \tag{7}$$

where $r_{\min}$ is the perihelion when the planet is closest to Sun and $r_{\max}$ is the aphelion when the planet is farthest from Sun. Show that the precession of perihelion is zero for the Kepler problem.

When a small correction

$$\delta U(r) = \frac{\beta}{r^2} \tag{8}$$

is added to the potential energy the precession of the perihelion is no longer zero. Expanding in powers of δU determine the precession of the perihelion to the leading order for this correction.

3. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,

$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{9}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyse the addition of a subluminal speed and a superluminal speed? That is, is the 'sum' subluminal or superluminal. Is the answer unique?

4. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by the equation of a hyperbola

$$x^2 - c^2t^2 = x_0^2, \qquad x_0 = \frac{c^2}{\alpha}. \tag{10}$$

This is the motion of a particle 'dropped' from $x = x_0$ at $t = 0$ in region of constant (proper) acceleration. Will a photon dispatched to 'chase' this particle at $t = 0$ from $x = 0$ ever catch up with it? If yes, when and where does it catch up?