Midterm Exam No. 02 (Spring 2017)

PHYS 510: Classical Mechanics

Date: 2017 Apr 11

1. (**20 points.**) In terms of the Lagrangian function $L(\mathbf{r}, \mathbf{v}, t)$ the action $W[\mathbf{r}; t_1, t_2]$ is defined as

$$W[\mathbf{r}; t_1, t_2] = \int_{t_1}^{t_2} dt\, L(\mathbf{r}, \mathbf{v}, t), \tag{1}$$

where $\mathbf{v} = d\mathbf{r}/dt$. Find the change in the action under an infinitesimal general time transformation

$$\bar{t} = t - \delta t(t), \quad \delta t(t_1) = 0, \quad \delta t(t_1) = 0. \tag{2}$$

In particular, evaluate the functional derivative

$$\frac{\delta W}{\delta t(t)} \tag{3}$$

for the variation $\delta t(t)$ satisfying the constraints of Eq. (2).

2. (**20 points.**) Noether's theorem, in the context of rotational symmetry, states that if the Lagrangian does not change under an infinitesimal rigid rotation, then the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved. Prove that the converse of Noether's theorem is also true. For simplicity consider velocity independent potentials.

3. (**20 points.**) A (spherical) pendulum is suspended such that a mass m is able to move freely on the surface of a sphere of radius a (the length of the pendulum). The spherical pendulum is suitably described by the Lagrangian function

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}mv^2 - mgz + \frac{1}{2}\left(\frac{r^2}{a^2} - 1\right)\mathbf{T} \cdot \mathbf{r}, \tag{4}$$

where $\mathbf{r}$ is the position vector with center of sphere as origin and $\mathbf{v} = d\mathbf{r}/dt$. Assume the acceleration due to gravity is downward, such that $\mathbf{g} = -g\,\hat{\mathbf{z}}$. Derive an expression for $\mathbf{T}$. In particular, give the physical interpretation of $\mathbf{T}$.

4. (**20 points.**) Starting from the Lagrangian for the Kepler problem,

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}, \tag{5}$$

derive Kepler's first law of planetary motion, which states that the orbit of a planet is a conic section.