Homework No. 06 (Spring 2017)

PHYS 510: Classical Mechanics

Due date: 2017 Mar 21 (Tuesday) 4.30pm

1. (**40 points.**) Consider the function describing a paraboloid
$$f(x,y) = a(x^2 + y^2). \tag{1}$$
A straight line on the xy plane, $y = mx + c$, can be interpreted as a condition of constraint
$$g(x,y) = y - mx - c = 0. \tag{2}$$
Let us determine the point on the line where the function $f(x,y)$ has an extremum value.

   (a) Construct the function
$$F(x) = f(x, mx + c). \tag{3}$$
Using the extremum principle, $dF/dx = 0$, show that the point on the line where the function f is an extremum is
$$x = -\frac{mc}{1+m^2}, \quad y = \frac{c}{1+m^2}. \tag{4}$$

   (b) Construct the function
$$h(x,y) = f(x,y) + \lambda g(x,y). \tag{5}$$
Evaluate $\boldsymbol{\nabla} h$, $\boldsymbol{\nabla} f$, and $\boldsymbol{\nabla} g$. Show that $\boldsymbol{\nabla} h = 0$ implies
$$x = \frac{\lambda m}{2a}, \quad y = -\frac{\lambda}{2a}. \tag{6}$$
Use this in the condition of constraint to derive
$$\lambda = -\frac{2ac}{1+m^2}. \tag{7}$$
Use the above expression for λ in Eq. (6) to find the point on the line where the function f is an extremum.

2. (**50 points.**) Spherical pendulum: Consider a pendulum that is suspended such that a mass m is able to move freely on the surface of a sphere of radius a (the length of the pendulum). The mass is then subject to the condition of constraint
$$F = \frac{1}{2}(x^2 + y^2 + z^2 - a^2) = 0, \tag{8}$$
where the factor of 1/2 is introduced anticipating cancellations. Consider the Lagrangian function
$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m \dot{\mathbf{r}}^2 - mgz - \lambda F. \tag{9}$$

(a) Evaluate the gradient $\boldsymbol{\nabla}$ of the condition of constraint. Show that

$$\boldsymbol{\nabla} F = \mathbf{r}. \tag{10}$$

(b) Using the Euler-Lagrange equations derive the equations of motion

$$m\ddot{\mathbf{r}} = -mg\hat{\mathbf{z}} + \lambda\mathbf{r}. \tag{11}$$

(c) Derive an expression for λ. In particular, show that it can be expressed in the form

$$\lambda a = \hat{\mathbf{r}} \cdot \mathbf{N}. \tag{12}$$

Find $\mathbf{N}$. Give the physical interpretation of $\mathbf{N}$ using D'Alembert's principle.

(d) Show that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = m\dot{\mathbf{r}}$ is the momentum of the particle, about the z-axis is conserved. That is,

$$\frac{d}{dt}(\hat{\mathbf{z}} \cdot \mathbf{L}) = 0. \tag{13}$$

Show that this also implies the conservation of the areal velocity

$$\frac{dS}{dt} = \frac{1}{2}(x\dot{y} - y\dot{x}), \tag{14}$$

where S is the area swept out.

(e) Show that

$$\frac{dF}{dt} = \mathbf{r} \cdot \dot{\mathbf{r}} = 0. \tag{15}$$

Using this derive the statement of conservation of energy,

$$\frac{dH}{dt} = 0, \qquad H = \frac{1}{2}m\dot{\mathbf{r}}^2 + mgz, \tag{16}$$

starting from the equation of motion in Eq. (11) and multiplying by $\dot{\mathbf{r}}$.