Homework No. 07 (Spring 2017)

PHYS 510: Classical Mechanics

Due date: 2017 Mar 28 (Tuesday) 4.30pm

1. (**40 points.**) In terms of the Lagrangian function $L(\mathbf{r}, \mathbf{v}, t)$ the action functional $W[\mathbf{r}; t_1, t_2]$ is defined as

$$W[\mathbf{r}; t_1, t_2] = \int_{t_1}^{t_2} dt\, L(\mathbf{r}, \mathbf{v}, t), \tag{1}$$

where $\mathbf{v} = d\mathbf{r}/dt$. For arbitrary infinitesimal coordinate variations

$$\bar{\mathbf{r}}(t) = \mathbf{r}(t) - \delta\mathbf{r}(t), \tag{2}$$

and infinitesimal general time transformation

$$\bar{t} = t - \delta t(t), \tag{3}$$

the change in action is given by

$$\begin{aligned} \delta W =& \int_{t_1}^{t_2} dt \frac{d}{dt}\Big[\mathbf{p}\cdot\delta\mathbf{r} - H\delta t\Big] \\ &+ \int_{t_1}^{t_2} dt \left[\delta t \left(\frac{dH}{dt} + \frac{\partial L}{\partial t}\right) + \delta\mathbf{r}\cdot\left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}}\right)\right], \end{aligned} \tag{4}$$

where the canonical momentum and the Hamiltonian are defined as

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} \quad \text{and} \quad H = \mathbf{v}\cdot\mathbf{p} - L \tag{5}$$

respectively. In terms of the Hamiltonian the action takes the form

$$W[\mathbf{r}, \mathbf{p}; t_1, t_2] = \int_{t_1}^{t_2} dt \Big[\mathbf{v}\cdot\mathbf{p} - H(\mathbf{r}, \mathbf{p}, t)\Big]. \tag{6}$$

Show that for for arbitrary infinitesimal coordinate and momentum variations

$$\bar{\mathbf{r}}(t) = \mathbf{r}(t) - \delta\mathbf{r}(t) \quad \text{and} \quad \bar{\mathbf{p}}(t) = \mathbf{p}(t) - \delta\mathbf{p}(t), \tag{7}$$

and infinitesimal general time transformation, the change in action is given by

$$\begin{aligned} \delta W =& \int_{t_1}^{t_2} dt \frac{d}{dt}\Big[\mathbf{p}\cdot\delta\mathbf{r} - H\delta t\Big] \\ &+ \int_{t_1}^{t_2} dt \left[\delta t \left(\frac{dH}{dt} - \frac{\partial H}{\partial t}\right) - \delta\mathbf{r}\cdot\left(\frac{d\mathbf{p}}{dt} + \frac{\partial H}{\partial \mathbf{r}}\right) + \delta\mathbf{p}\cdot\left(\frac{d\mathbf{r}}{dt} - \frac{\partial H}{\partial \mathbf{p}}\right)\right]. \end{aligned} \tag{8}$$

2. (**20 points.**) Consider infinitesimal rigid rotation, described by

$$\delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}, \quad \delta\mathbf{p} = \delta\boldsymbol{\omega} \times \mathbf{p}, \quad \delta t = 0, \tag{9}$$

where $d\delta\boldsymbol{\omega}/dt = 0$. Show that the variation in the action under the above rotation is

$$\frac{\delta W}{\delta\boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}}\right] = \mathbf{L}(t_2) - \mathbf{L}(t_1), \tag{10}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum. Thus, state the conditions to be satisfied by the Hamitonian for the angular momentum $\mathbf{L}$ of the system to be conserved.