Homework No. 08 (Spring 2017)

PHYS 510: Classical Mechanics

Due date: 2017 Apr 6 (Thursday) 4.30pm

1. (**40 points.**) The Hamiltonian for a Kepler problem is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{\alpha}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two constituent particles of masses m_1 and m_2.

(a) Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2}, \tag{2}$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{\alpha}{r}, \tag{3}$$

where

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{4}$$

(b) Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\alpha\mathbf{r}}{r^3}. \tag{5}$$

(c) Verify that the Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times \mathbf{L}}{\mu\alpha}, \tag{6}$$

are the three constants of motion for the Kepler problem.

2. (**50 points.**) (Refer Landau and Lifshitz, Problem 1 in Chapter 3.)
A simple pendulum, consisting of a particle of mass m suspended by a string of length l in a uniform gravitational field g, is described by the Hamiltonian

$$H = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi. \tag{7}$$

(a) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that

$$\frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0. \tag{8}$$

Thus, derive

$$\frac{dt}{T_0} = \frac{1}{2\pi}\frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{9}$$

where $T_0 = 2\pi\sqrt{l/g}$.

(b) Determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0\,\frac{2}{\pi}K\left(\sin\frac{\phi_0}{2}\right), \tag{10}$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \tag{11}$$

is the complete elliptic integral of the first kind.

(c) Using the power series expansion

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} \tag{12}$$

show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0\left[1 + \frac{\phi_0^2}{16} + \dots\right]. \tag{13}$$

(d) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60^\circ$.

(e) Plot the time period T as a function of ϕ_0. What can you conclude about the time period for $\phi_0 = \pi$?