Homework No. 10 (Spring 2017)

PHYS 510: Classical Mechanics

Due date: 2017 May 4 (Thursday) 4.30pm

1. (**20 points.**) The Lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}. \tag{1}$$

   (a) Evaluate γ for $v = 30\,\text{m/s}$ ($\sim 70\,\text{miles/hour}$).

   (b) Evaluate γ for $v = 3c/5$.

2. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{2}$$

respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

respectively. For infinitesimal transformations, $\beta_i = \delta\beta_i$ and $\omega_i = \delta\omega_i$ use the approximations

$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \delta\omega_i, \tag{4}$$

to identify the generator for boosts $\mathbf{N}$, and the generator for rotations the angular momentum $\mathbf{J}$,

$$\mathbf{L} = \mathbf{1} + \delta\boldsymbol{\beta}\cdot\mathbf{N} \qquad \text{and} \qquad \mathbf{R} = \mathbf{1} + \delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{5}$$

respectively. Then derive

$$\Big[N_1, N_2\Big] = N_1N_2 - N_2N_1 = J_3. \tag{6}$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

3. (**60 points.**) The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}, \tag{7}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Jerzy Kocik, from the department of Mathematics in SIUC, has invented a geometric diagram that allows one to visualize the Poincaré formula. (Refer [1].) An interactive applet for exploring velocity addition is available at Kocik's web page [2]. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.)

   (a) Analyse what is obtained if you add two subluminal speeds?

   (b) Analyse what is obtained if you add a subluminal speed to speed of light?

   (c) Analyse what is obtained if you add a subluminal speed to a superluminal speed?

   (d) Analyse what is obtained if you add speed of light to another speed of light?

   (e) Analyse what is obtained if you add a superluminal speed to speed of light?

   (f) Analyse what is obtained if you add two superluminal speeds?

4. (**30 points.**) Let

$$\tanh\theta = \beta, \tag{8}$$

where $\beta = v/c$. Addition of (parallel) velocities in terms of the parameter θ obeys the arithmatic addition

$$\theta = \theta_a + \theta_b. \tag{9}$$

   (a) Invert the expression in Eq. (8) to find the explicit form of θ in terms of β as a logarithm.

   (b) Show that Eq. (9) leads to the relation

$$\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{1+\beta_a}{1-\beta_a}\right)\left(\frac{1+\beta_b}{1-\beta_b}\right). \tag{10}$$

   (c) Using Eq. (10) derive the Poincaré formula for the addition of (parallel) velocities.

5. (**20 points.**) (Refer Hughston and Tod's book.)
Prove that

   (a) if p_μ is a time-like vector and $p^\mu s_\mu = 0$ then s^μ is necessarily space-like.

   (b) if p_μ and q^μ are both time-like vectors and $p^\mu q_\mu > 0$ then either both are future-pointing or both are past-pointing.

6. (**100 points.**) Eigenvalues of the energy momentum tensor. (We choose $c = 1$, which is easily undone by replacing $\mathbf{E} \to \mathbf{E}/c$ everywhere.)

(a) Using

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{11}$$

and

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{12}$$

evaluate the following:

i. $F^{\mu\lambda}F_{\lambda\nu}$

ii. $\tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu}$

iii. Then, derive

$$F^{\mu\lambda}\tilde{F}_{\lambda\nu} = \delta^{\mu}{}_{\nu}\mathbf{E}\cdot\mathbf{B}, \tag{13a}$$

$$\tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu} - F^{\mu\lambda}F_{\lambda\nu} = \delta^{\mu}{}_{\nu}(B^2 - E^2). \tag{13b}$$

(b) Define

$$2f = (B^2 - E^2) \quad \text{and} \quad g = \mathbf{E}\cdot\mathbf{B}. \tag{14}$$

Thus, construct matrix (or dyadic) equations

$$\mathbf{F}\cdot\tilde{\mathbf{F}} = g\mathbf{1}, \tag{15a}$$

$$\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}} - \mathbf{F}\cdot\mathbf{F} = 2f\mathbf{1}, \tag{15b}$$

in terms of matrices (or dyadics) $\mathbf{F}$ and $\tilde{\mathbf{F}}$.

(c) Show that the eigenvalues λ of the field tensor $\mathbf{F}$ satisfy the quartic equation

$$\lambda^4 + 2f\lambda^2 - g^2 = 0. \tag{16}$$

(d) Evaluate the eigenvalues to be $\pm\lambda_1$ and $\pm\lambda_2$ where

$$\lambda_1 = \sqrt{-f - \sqrt{f^2 + g^2}} = \frac{i}{\sqrt{2}}\left[\sqrt{f + ig} + \sqrt{f - ig}\right], \tag{17}$$

$$\lambda_2 = \sqrt{-f + \sqrt{f^2 + g^2}} = \frac{i}{\sqrt{2}}\left[\sqrt{f + ig} - \sqrt{f - ig}\right]. \tag{18}$$

(e) Show that

i. if $B^2 - E^2 = 0$, then the eigenvalues are $\pm\sqrt{g}$ and $\pm i\sqrt{g}$.

ii. if $\mathbf{B}\cdot\mathbf{E} = 0$, then the eigenvalues are 0, 0, and $\pm\sqrt{2f}$.

(f) Prove the following:

i. There is no Lorentz transformation connecting two reference frames such that the field is purely magnetic in origin in one and purely electric in origin in the other.

ii. If $B^2 - E^2 > 0$ in a frame, then there exists a frame in which the field is purely magnetic.

iii. If $B^2 - E^2 < 0$ in a frame, then there exists a frame in which the field is purely electric.

iv. If $B^2 - E^2 = 0$ in a frame, then there exists a frame in which

— $\mathbf{B}$ is perpendicular to $\mathbf{E}$, if $\mathbf{B} \cdot \mathbf{E} = 0$.

— $\mathbf{B}$ is parallel to $\mathbf{E}$, if $\mathbf{B} \cdot \mathbf{E} > 0$.

— $\mathbf{B}$ is antiparallel to $\mathbf{E}$, if $\mathbf{B} \cdot \mathbf{E} < 0$.

References

[1] J. Kocik. Geometric diagram for relativistic addition of velocities. *Am. J. Phys.*, 80:737–739, August 2012.

[2] J. Kocik. An interactive applet for exploring relativistic velocity addition. http://lagrange.math.siu.edu/Kocik/relativity/diagram.html.