

Midterm Exam No. 01 (Fall 2017)

PHYS 440: Quantum Mechanics

Date: 2017 Sep 19

1. (100 points.) Consider a normalized state

$$|\rangle = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 \\ y_1 + iy_2 \end{pmatrix}, \quad |u|^2 + |v|^2 = 1. \quad (1)$$

- (a) Show that the expectation value of the Pauli matrices with respect to the above state satisfies

$$\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 = 1. \quad (2)$$

Thus, conclude that the magnitude of the expectation value of each of the Pauli matrix is less than or equal to unity.

- (b) Define the errors in the measurement of the Pauli matrices to be

$$(\delta \sigma'_x)^2 = \langle |\{\sigma_x - \langle \sigma_x \rangle\}^2| \rangle, \quad (3a)$$

$$(\delta \sigma'_y)^2 = \langle |\{\sigma_y - \langle \sigma_y \rangle\}^2| \rangle, \quad (3b)$$

$$(\delta \sigma'_z)^2 = \langle |\{\sigma_z - \langle \sigma_z \rangle\}^2| \rangle. \quad (3c)$$

Show that

$$(\delta \sigma'_x)^2 = 1 - \langle \sigma_x \rangle^2, \quad (4a)$$

$$(\delta \sigma'_y)^2 = 1 - \langle \sigma_y \rangle^2, \quad (4b)$$

$$(\delta \sigma'_z)^2 = 1 - \langle \sigma_z \rangle^2. \quad (4c)$$

Thus, conclude that the errors in the measurement of each of the Pauli matrix is less than or equal to unity. Show that

$$(\delta \sigma'_x)^2 + (\delta \sigma'_y)^2 + (\delta \sigma'_z)^2 = 2. \quad (5)$$

- (c) Using Robertson's generalization of Heisenberg's uncertainty relation

$$(\delta A)(\delta B) \geq \frac{1}{2} |\langle C \rangle|, \quad (6)$$

deduce the uncertainty relations for the Pauli matrices to be

$$(\delta \sigma'_x)^2 (\delta \sigma'_y)^2 \geq \langle \sigma_z \rangle^2, \quad (7a)$$

$$(\delta \sigma'_y)^2 (\delta \sigma'_z)^2 \geq \langle \sigma_x \rangle^2, \quad (7b)$$

$$(\delta \sigma'_z)^2 (\delta \sigma'_x)^2 \geq \langle \sigma_y \rangle^2. \quad (7c)$$

Combine these uncertainty relations to derive an uncertainty relation involving all the three Pauli matrices,

$$(\delta\sigma'_x)^2(\delta\sigma'_y)^2 + (\delta\sigma'_y)^2(\delta\sigma'_z)^2 + (\delta\sigma'_z)^2(\delta\sigma'_x)^2 \geq 1. \quad (8)$$

(d) Show that minimal uncertainty in each of the above relations is attained, if

$$\langle\sigma_x\rangle\langle\sigma_y\rangle = 0, \quad (9a)$$

$$\langle\sigma_y\rangle\langle\sigma_z\rangle = 0, \quad (9b)$$

$$\langle\sigma_z\rangle\langle\sigma_x\rangle = 0. \quad (9c)$$

Since the expectation value of the Pauli matrices satisfy Eq. (2) all three of the expectation value of Pauli matrices can not be zero for a given state. But, two of them can be zero, which is sufficient for minimizing all the above uncertainty relations. For example, a state satisfying

$$\langle\sigma_x\rangle = \langle\sigma_y\rangle = 0 \quad (10)$$

minimizes all the four uncertainty relations above.

(e) Show that

$$|\text{min}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (11)$$

is such a state. Evaluate $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, $\langle\sigma_z\rangle$, $\langle\sigma_x^2\rangle$, $\langle\sigma_y^2\rangle$, $\langle\sigma_z^2\rangle$, $\delta\sigma'_x$, $\delta\sigma'_y$, and $\delta\sigma'_z$, when the system is in this state.

(f) Understand the Stern-Gerlach experiment in this light.