Homework No. 03 (Fall 2017)

PHYS 440: Quantum Mechanics

Due date: 2017 Sep 14 (Thursday) 4.30pm

1. (**20 points.**) A unitary matrix is defined by
$$U^\dagger U = 1, \tag{1}$$
where $\dagger$ stands for transpose and complex conjugation. Show that
$$U = e^{iH} \tag{2}$$
is unitary if H is Hermitian, that is $H^\dagger = H$.

2. (**20 points.**) Prove that Hermitian operators have real eigenvalues. Further, show that any two eigenfunctions belonging to distinct (unequal) eigenvalues of a Hermitian operator are mutually orthogonal.

3. (**20 points.**) Show that the combination $X^\dagger X$ is Hermitian, irrespective of X being Hermitian. Use this to deduce that the eigenvalues of $X^\dagger X$ are non-negative.

4. (**60 points.**) Let A and B be Hermitian operators.

   (a) Consider the expectation or average values of A and B in the physical state $|\ \rangle$,
   $$\langle A\rangle = \langle\ |A|\ \rangle, \qquad \langle B\rangle = \langle\ |B|\ \rangle, \tag{3}$$
   and the mean square deviation from these averages,
   $$(\delta A)^2 = \langle\ |(A - \langle A\rangle)^2|\ \rangle \equiv \langle 1|1\rangle, \tag{4}$$
   $$(\delta B)^2 = \langle\ |(B - \langle B\rangle)^2|\ \rangle \equiv \langle 2|2\rangle, \tag{5}$$
   where
   $$\langle 1| = \langle\ |(A - \langle A\rangle), \qquad |2\rangle = (B - \langle B\rangle)|\ \rangle, \tag{6}$$

   (b) (Prove the Schwarz inequality.) Use the Schwarz inequality to learn
   $$(\delta A)^2(\delta B)^2 = \langle 1|1\rangle\langle 2|2\rangle \geq |\langle 1|2\rangle|^2, \tag{7}$$
   where the equal sign applies only when $|1\rangle$ is parallel to $|2\rangle$.

(c) Show that the antisymmetric product of two Hermitian operators X and Y,

$$C = \frac{1}{i}(XY - YX) = \frac{1}{i}\big[X, Y\big], \tag{8}$$

is also Hermitian, that is, $C^\dagger = C$. Further, show that the symmetric construction,

$$(XY + YX) = \big\{X, Y\big\}, \tag{9}$$

is also Hermitian. Thus, the product XY, which is not Hermitian, can be expressed as a combination of two Hermitian operators,

$$XY = \frac{1}{2}(XY + YX) + \frac{i}{2}C. \tag{10}$$

Remember that the expectation values of Hermitian operators are real.

(d) Let

$$X = A - \langle A \rangle, \qquad Y = B - \langle B \rangle. \tag{11}$$

Thus, derive

$$|\langle\, |(XY)|\, \rangle|^2 = \frac{1}{4}|\langle\, |(XY + YX)|\, \rangle|^2 + \frac{1}{4}|\langle\, |C|\, \rangle|^2. \tag{12}$$

(e) Using Eq. (12) in Eq. (7) derive Robertson's generalization of Heisenberg's uncertainty relation

$$(\delta A)(\delta B) \geq \frac{1}{2}|\langle C \rangle|. \tag{13}$$

(f) Apply this to the pairs $(A, B) = (q, p)$ and $(A, B) = (\sigma_x, \sigma_y)$.