

Homework No. 05 (Fall 2017)

PHYS 440: Quantum Mechanics

Due date: 2017 Oct 5 (Thursday) 4.30pm

1. Any system obeying the Newtonian equation of motion

$$\frac{dp}{dt} = -kx = -\frac{\partial}{\partial x} \left(\frac{1}{2} kx^2 \right) \quad (1)$$

is broadly termed a simple harmonic oscillator.

2. A harmonic oscillator is also described by the Hamiltonian

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2} kx^2. \quad (2)$$

3. We upgrade to quantum mechanics from the classical picture by imposing the commutation relations

$$[x, x] = 0, \quad [p, p] = 0, \quad [x, p] = i\hbar, \quad (x^\dagger = x, \quad p^\dagger = p,) \quad (3)$$

on the dynamical variables x and p . Further, the Hamilton equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad (4a)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \quad (4b)$$

are generalized to include the Heisenberg equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{1}{i\hbar} [x, H], \quad (5a)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = \frac{1}{i\hbar} [p, H]. \quad (5b)$$

Evaluate the Hamilton and Heisenberg equations of motion for the harmonic oscillator. Further, by projecting the Hamiltonian operator on an arbitrary state $|\rangle$, and then projecting the resultant state on a position eigenstate $\langle x'|$, and identifying the expectation value of the Hamiltonian as the energy E of the system in state $|\rangle$,

$$\langle x'|H|\rangle = E\langle x'|\rangle, \quad (6)$$

derive the time-independent Schrödinger equation for the harmonic oscillator.

4. In terms of the non-Hermitian operators, $\omega = \sqrt{k/m}$,

$$y = \frac{1}{\sqrt{2\hbar\omega}} \left(\sqrt{k} x + i \frac{p}{\sqrt{m}} \right), \quad (7a)$$

$$y^\dagger = \frac{1}{\sqrt{2\hbar\omega}} \left(\sqrt{k} x - i \frac{p}{\sqrt{m}} \right), \quad (7b)$$

verify that

$$[y, y^\dagger] = 1, \quad [y, y] = 0, \quad [y^\dagger, y^\dagger] = 0. \quad (8)$$

Show that the Hamiltonian for a harmonic oscillator is given by

$$H(y, y^\dagger) = \hbar\omega \left(y^\dagger y + \frac{1}{2} \right). \quad (9)$$

Evaluate the content of the Heisenberg equations of motion

$$\frac{dy}{dt} = \frac{1}{i\hbar} [y, H], \quad (10a)$$

$$\frac{dy^\dagger}{dt} = \frac{1}{i\hbar} [y^\dagger, H]. \quad (10b)$$

5. The Hamiltonian of a harmonic oscillator is given in terms of the operator $y^\dagger y$. Thus, we shall be interested in finding the eigenvalues and eigenvectors of this operator. In particular, our goal will be to solve the eigenvalue equation

$$y^\dagger y |n'\rangle = n' |n'\rangle, \quad (11)$$

where $|n'\rangle$ are the eigenvectors corresponding to the eigenvalues n' .

- (a) Argue that $y^\dagger y$ can not be represented using finite dimensional matrices. Thus, argue that the eigenvalues n' must be infinite of them.
- (b) Prove that $y^\dagger y$ is an Hermitian operator. Thus, the eigenvalues n' must be real. Further, deduce that the eigenvalues n' are non-negative.
- (c) Show that y is a lowering operator, that is,

$$y |n'\rangle = c_{n'} |n' - 1\rangle, \quad (12)$$

where $c_{n'}$ is to be determined. Thus, prove that, if n' is an eigenvalue, then $(n' - 1)$ is also an eigenvalue.

- (d) Show that y^\dagger is a raising operator, that is,

$$y^\dagger |n'\rangle = d_{n'} |n' + 1\rangle, \quad (13)$$

where $d_{n'}$ is to be determined. Thus, prove that, if n' is an eigenvalue, then $(n' + 1)$ is also an eigenvalue.

- (e) Together, to satisfy the requirement that the eigenvalues are non-negative, the implication is that the lowering operator can not indefinitely lower the state. Thus, argue the existence of the ground eigenstate that satisfies

$$y|0\rangle = 0. \quad (14)$$

Thus, learn that $n' = 0$ is an eigenstate.

6. (**50 points.**) To determine $d_{n'}$ we construct an eigenstate in terms of the ground state,

$$|n'\rangle = \frac{(y^\dagger)^{n'}}{D_{n'}}|0\rangle, \quad D_{n'} = d_0 d_1 \dots d_{n'-1}. \quad (15)$$

Presuming the eigenstates are normalized, use

$$\langle m'|n'\rangle = \delta_{m'n'} \quad (16)$$

to learn that

$$|D_{n'}|^2 = \langle 0|y^{n'}(y^\dagger)^{n'}|0\rangle. \quad (17)$$

Show that

$$\langle 0|y^{n'}(y^\dagger)^{n'}|0\rangle = n' \langle 0|y^{n'-1}(y^\dagger)^{n'-1}|0\rangle. \quad (18)$$

(Hint: Show that $[y, (y^\dagger)^{n'}] = n'(y^\dagger)^{n'-1}$.) Thus, deduce that $|D_{n'}| = \sqrt{n'!}$ and $|d_{n'}| = \sqrt{n'+1}$. Thus, we have

$$y^\dagger|n'\rangle = \sqrt{n'+1}|n'+1\rangle. \quad (19)$$

Operate the lowering operator y on both sides of the above equation and decipher the statement

$$y|n'\rangle = \sqrt{n'}|n'-1\rangle \quad (20)$$

to learn that $c_{n'} = \sqrt{n'}$.

7. The projection of the eigenstates $|n'\rangle$ on the position eigenstates $\langle x'|$ leads to the construction of the corresponding wavefunctions

$$\psi_{n'}(x') = \langle x'|n'\rangle. \quad (21)$$

We will set $k = m = 1$ by suitable scaling of x and p and further set $\hbar = 1$ to avoid clutter in the equations. We shall also drop the primes to represent the eigenvalues.

- (a) Starting from Eq. (14) deduce the differential equation satisfied by the ground state to be

$$\left(x + \frac{\partial}{\partial x}\right)\psi_0(x) = 0. \quad (22)$$

Thus, show that

$$\psi_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2}. \quad (23)$$

(b) Starting from the statement in Eq (15),

$$|n\rangle = \frac{(y^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad (24)$$

deduce the relation

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(x - \frac{\partial}{\partial x} \right)^n \psi_0(x). \quad (25)$$

(c) Verify that

$$\left(x - \frac{\partial}{\partial x} \right) f(x) = \left(-e^{\frac{1}{2}x^2} \frac{\partial}{\partial x} e^{-\frac{1}{2}x^2} \right) f(x). \quad (26)$$

Thus, show that

$$\left(x - \frac{\partial}{\partial x} \right)^n f(x) = e^{\frac{1}{2}x^2} \left(-\frac{\partial}{\partial x} \right)^n e^{-\frac{1}{2}x^2} f(x). \quad (27)$$

(d) The Hermite polynomials are defined as

$$H_n(x) = e^{x^2} \left(-\frac{\partial}{\partial x} \right)^n e^{-x^2}. \quad (28)$$

Evaluate $H_n(x)$ for $n = 0, 1, 2, 3, 4$. Thus, derive

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \psi_0(x) H_n(x). \quad (29)$$

Plot $\psi_n(x)$ for $n = 0, 1, 2, 3, 4$.