Homework No. 06 (Fall 2017)

PHYS 440: Quantum Mechanics

Due date: 2017 Oct 12 (Thursday) 4.30pm

1. (**40 points.**) A particle of mass m and charge q moving in a uniform magnetic field $\mathbf{B}$ experiences a velocity dependent force $\mathbf{F}$ given by the expression

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}, \tag{1}$$

where $\mathbf{v}(t) = d\mathbf{x}/dt$ is the velocity of the particle in terms of its position $\mathbf{x}(t)$. Choose the magnetic field to be along the positive z direction, $\mathbf{B} = B\hat{\mathbf{z}}$.

   (a) For the case when the particle starts at rest at the origin at time $t = 0$, use the initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + v_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{2}$$
$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{3}$$

   to solve the differential equation in Eq. (1) to find the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ as a function of time. Use $\omega = qB/m$.

   (b) In particular, prove that the particle takes a circular path. What is the radius of the circle? Determine the coordinates of the center of the circle?

2. (**20 points.**) A homogeneous magnetic field $\mathbf{B}$ is characterized by the vector potential

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}. \tag{4}$$

   (a) Evaluate $\boldsymbol{\nabla} \times \mathbf{A}$. (Hint: $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$.)

   (b) Evaluate $\boldsymbol{\nabla} \cdot \mathbf{A}$.

   (c) Is this construction unique? (Hint: Remember the freedom of gauge transformation.)

   (d) Now, for the case of $\mathbf{B} = (0, 0, B)$, pointing in the z direction, show that $\mathbf{A} = (0, Bx, 0)$ is a solution. Find another solution.

3. (**20 points.**) The components of $\boldsymbol{\sigma}$ are Pauli matrices and satisfy the commutation relations (of spin)

$$\big[\sigma_i, \sigma_j\big] = 2i\varepsilon_{ijk}\sigma_k. \tag{5}$$

The components of $\boldsymbol{\tau}$ are also Pauli matrices and satisfy the commutation relations (of isospin)

$$\big[\tau_i, \tau_j\big] = 2i\varepsilon_{ijk}\tau_k. \tag{6}$$

The operators $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ act on disjoint Hilbert spaces. Find the eigenvalues and eigenvectors of the operator construction

$$H = \sigma_z + \tau_z, \tag{7}$$

which is a short hand for the expression

$$H = \sigma_z \otimes 1 + 1 \otimes \tau_z. \tag{8}$$