Homework No. 07 (Fall 2017)

PHYS 440: Quantum Mechanics

Due date: 2017 Oct 26 (Thursday) 4.30pm

1. (**40 points.**) Consider the operator constructions

$$\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) \quad \text{and} \quad \sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y). \tag{1}$$

   (a) Show that

$$\sigma_+|-\rangle = |+\rangle, \qquad \sigma_+|+\rangle = 0, \tag{2a}$$
$$\sigma_-|-\rangle = 0, \qquad \sigma_-|+\rangle = |-\rangle. \tag{2b}$$

   (b) Evaluate σ_+^2, σ_-^2, $\sigma_+\sigma_-$, $\sigma_-\sigma_+$, the commutation relation $\big[\sigma_+, \sigma_-\big]$, and the anti-commutation relation $\big\{\sigma_+, \sigma_-\big\}$.

2. (**40 points.**) Consider the operator

$$\gamma_0 = y\sigma_+ + y^\dagger\sigma_-, \tag{3}$$

   where $\sigma_+ = (\sigma_x + i\sigma_y)/2$ and $\sigma_- = (\sigma_x - i\sigma_y)/2$.

   (a) Let $|n', \pm 1\rangle$, $n' = 0, 1, 2, \ldots$, be the common eigenvectors of $y^\dagger y$ and σ_z. Consider the subspace spanned by the eigenvectors $|n', +1\rangle$ and $|n' + 1, -1\rangle$. Construct the matrix representation of the operator γ_0 in this subspace by evaluating the elements in

$$\gamma_0 = \begin{bmatrix} \langle n', +1|\gamma_0|n', +1\rangle & \langle n', +1|\gamma_0|n' + 1, -1\rangle \\ \langle n' + 1, -1|\gamma_0|n', +1\rangle & \langle n' + 1, -1|\gamma_0|n' + 1, -1\rangle \end{bmatrix}. \tag{4}$$

   (b) Find the eigenvalues of the operator γ_0.

   (c) Find the two normalized eigenvectors (upto a phase) of the operator γ_0 in the subspace discussed above, and express them in terms of the respective common eigenvectors.

3. (**40 points.**) Consider the operator

$$\gamma = y\sigma_+ + y^\dagger\sigma_- + \epsilon\sigma_z, \tag{5}$$

   where $\sigma_+ = (\sigma_x + i\sigma_y)/2$ and $\sigma_- = (\sigma_x - i\sigma_y)/2$.

(a) Let $|n', \pm 1\rangle$, $n' = 0, 1, 2, \ldots$, be the common eigenvectors of $y^\dagger y$ and σ_z. Consider the subspace spanned by the eigenvectors $|n', +1\rangle$ and $|n'+1, -1\rangle$. Construct the matrix representation of the operator γ in this subspace by evaluating the elements in

$$\gamma = \begin{bmatrix} \langle n', +1|\gamma|n', +1\rangle & \langle n', +1|\gamma|n'+1, -1\rangle \\ \langle n'+1, -1|\gamma|n', +1\rangle & \langle n'+1, -1|\gamma|n'+1, -1\rangle \end{bmatrix}. \tag{6}$$

(b) Find the eigenvalues of the operator γ.

(c) Find the two normalized eigenvectors (upto a phase) of the operator γ in the subspace discussed above, and express them in terms of the respective common eigenvectors. In particular, show that

$$|n' + \tfrac{1}{2}, +1\rangle = \cos\frac{\alpha_n}{2}|n', +1\rangle + \sin\frac{\alpha_n}{2}|n'+1, -1\rangle, \tag{7a}$$

$$|n' + \tfrac{1}{2}, -1\rangle = -\sin\frac{\alpha_n}{2}|n', +1\rangle + \cos\frac{\alpha_n}{2}|n'+1, -1\rangle, \tag{7b}$$

where

$$\tan\alpha_n = \frac{\sqrt{n'+1}}{\epsilon}. \tag{8}$$