

Homework No. 08 (Fall 2017)

PHYS 440: Quantum Mechanics

Due date: 2017 Nov 2 (Thursday) 4.30pm

1. **(30 points.)** Find the two roots of the quadratic equation

$$\varepsilon x^2 - x + 1 = 0. \quad (1)$$

Show that the series expansion of the roots in the variable ε are

$$x = \begin{cases} 1 + \varepsilon + \mathcal{O}(\varepsilon)^2, \\ \frac{1}{\varepsilon} - 1 + \mathcal{O}(\varepsilon). \end{cases} \quad (2)$$

Let us treat the quadratic equation in Eq. (1) as a perturbation to the trivial equation

$$x_0 = 1, \quad (3)$$

obtained by setting $\varepsilon = 0$ in Eq. (1). To find the roots to the leading order in the perturbation parameter ε we use the ansatz

$$x = x_0 + a_1\varepsilon + \mathcal{O}(\varepsilon)^2. \quad (4)$$

Substitute Eq. (4) in Eq. (1) to show that $a_1 = 1$. Thus, we learn that

$$x = 1 + \varepsilon + \mathcal{O}(\varepsilon)^2, \quad (5)$$

which is indeed one of the root, to the leading order in ε . What about the other root? To this end, let us make the change of variables using the substitution

$$x = \frac{y}{\varepsilon}. \quad (6)$$

Show that this leads to the quadratic equation

$$y^2 - y + \varepsilon = 0. \quad (7)$$

The unperturbed equation, obtained by setting $\varepsilon = 0$, is

$$y_0^2 - y_0 = 0, \quad (8)$$

whose roots are $y_0 = 0, 1$. Use the ansatz

$$y = y_0 + b_1\varepsilon + b_2\varepsilon^2 + \mathcal{O}(\varepsilon)^3 \quad (9)$$

to derive

$$b_1 = \frac{1}{1 - 2y_0}, \quad b_2 = \frac{b_1^2}{1 - 2y_0}. \quad (10)$$

Thus, reproduce the results in Eq. (2), this time using the techniques of perturbation theory. Using the ideas described above find the five roots of the quintic equation

$$\varepsilon x^5 - x + 1 = 0, \quad (11)$$

to the leading order in ε . (Hint: Try $x = y/\varepsilon^{\frac{1}{4}}$.) Verify your solutions for $\varepsilon = 0.001$, say using Mathematica. Abel's impossibility theorem rules out algebraic solutions to general quintic equations.

2. **(20 points.)** Consider the unperturbed Hamiltonian to be

$$H = J_z, \quad (12)$$

where \mathbf{J} is the angular momentum. Let $j = 1/2$. Consider the perturbation

$$H + \delta H = J_z + \alpha J_x. \quad (13)$$

Determine the eigenvalues and eigenfunctions of $H + \delta H$ to the leading order in α . Further, verify that the eigenfunctions are orthonormal upto the leading order in α .