Homework No. 09 (Fall 2017)

PHYS 440: Quantum Mechanics

Due date: 2017 Nov 9 (Thursday) 4.30pm

1. (**Extra Credit.**) The components of angular momentum $\mathbf{J}$ satisfy the commutation relations

$$\frac{1}{i\hbar}\big[J_i, J_j\big] = \varepsilon_{ijk} J_k. \tag{1}$$

The general properties of angular momentum can be deduced from these commutation relations. Since $\mathbf{J}^2$ is a scalar, it commutes with angular momentum $\mathbf{J}$. Thus, the common eigenvectors of $\mathbf{J}^2$ and J_z constitute a suitable set of basis vectors for discussing a dynamical system involving only the angular momentum. Let us denote the eigenvalues of these operators by the labeling scheme $\mathbf{J}'^2 = j(j+1)\hbar^2$, and $J_z{}' = m\hbar$. Thus, we write

$$\frac{1}{\hbar^2}\mathbf{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \tag{2a}$$

$$\frac{1}{\hbar}J_z|j,m\rangle = m|j,m\rangle. \tag{2b}$$

Let us also construct (non-Hermitian) operators

$$J_\pm = J_x \pm iJ_y. \tag{3}$$

Observe that $J_\pm^\dagger = J_\mp$.

(a) Show that

$$\frac{1}{\hbar}J_z\Big\{J_+|j,m\rangle\Big\} = (m+1)\Big\{J_+|j,m\rangle\Big\}. \tag{4}$$

Thus deduce that if m is an eigenvalue of J_z, then $(m+1)$ is also an eigenvlaue of J_z. Similarly, show that

$$\frac{1}{\hbar}J_z\Big\{J_-|j,m\rangle\Big\} = (m-1)\Big\{J_-|j,m\rangle\Big\}. \tag{5}$$

Thus deduce that if m is an eigenvalue of J_z, then $(m-1)$ is also an eigenvlaue of J_z.

(b) Show that

$$J_+J_- = \mathbf{J}^2 - J_z^2 + \hbar J_z \tag{6}$$

is a Hermitian operator. A Hermitian operator has real eigenvalues, but, since $J_+J_- = J_-^\dagger J_-$, infer further that it has non-negative eigenvalues. Thus, deduce that

$$j(j+1) - m(m-1) \geq 0, \tag{7}$$

and then infer

$$-j \leq m \leq j+1. \tag{8}$$

Similarly, show how that

$$J_- J_+ = \mathbf{J}^2 - J_z^2 - \hbar J_z \tag{9}$$

is a Hermitian operator, and deduce that

$$j(j+1) - m(m+1) \geq 0, \tag{10}$$

and then infer

$$-j-1 \leq m \leq j. \tag{11}$$

Using Eqs. (8) and (11) in conjunction, show that

$$-j \leq m \leq j. \tag{12}$$

(Refer Sec. 36 Dirac's QM book.)

(c) Using Eq. (12) infer that

$$j \geq 0. \tag{13}$$

Note that $\mathbf{J}^2$ being the square of a Hermitian operator implies $j(j+1) \geq 0$, but it does not imply that j should be non-negative.

(d) The $2j$ transitions from $m = -j$ to $m = j$ happen in $n = 0, 1, 2, \ldots$ steps. Thus, $2j = n$. Thus, conclude that

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \tag{14a}$$

$$m = -j, -j+1, \ldots, j. \tag{14b}$$

(e) Repeat the above analysis starting from the labeling scheme

$$\mathbf{J}'^2 = \beta\hbar^2, \quad \text{and} \quad J_z{}' = m\hbar. \tag{15}$$

2. (**50 points.**) For $j = 1$:

(a) Determine the matrix representantion for

$$J_z, J_x, J_y, J_+, J_-, \text{and } J^2. \tag{16}$$

For example,

$$J_z = \begin{bmatrix} \langle j,j|J_z|j,j\rangle & \langle j,j|J_z|j,j-1\rangle & \cdots & \langle j,j|J_z|j,-j\rangle \\ \langle j,j-1|J_z|j,j\rangle & \langle j,j-1|J_z|j,j-1\rangle & \cdots & \langle j,j-1|J_z|j,-j\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle j,-j|J_z|j,j\rangle & \langle j,-j|J_z|j,j-1\rangle & \cdots & \langle j,-j|J_z|j,-j\rangle \end{bmatrix}. \tag{17}$$

(b) Evaluate

$$\text{Tr}(J_k), \quad \text{Tr}(J_k J_l), \quad \text{and} \quad \text{Tr}(J_k^2 J_l^2), \quad \text{for} \quad k,l = x,y,z. \tag{18}$$