Homework No. 11 (Fall 2017)

PHYS 440: Quantum Mechanics

Due date: 2017 Dec 7 (Thursday) 4.30pm

1. (**20 points.**) The polar equation of a conic of eccentricity ε is

$$r = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta}, \tag{1}$$

where $2a$ is the major-axis of the ellipse. The directrix of an ellipse is a line perpendicular to the major-axis at a distance d from the focus (origin). For an ellipse, the ratio between the radial distance of a point on the ellipse from the origin, and the distance of the point from the directrix, is the eccentricity. Thus, determine d in terms of a and ε.

2. (**20 points.**) The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Verify the following:

   (a) $\mathbf{r}\times\mathbf{p} + \mathbf{p}\times\mathbf{r} = 0$.

   (b) $\mathbf{r}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{r} = 3i\hbar$.

   (c) $(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{p}) - (\mathbf{b}\cdot\mathbf{p})(\mathbf{a}\cdot\mathbf{r}) = i\hbar(\mathbf{a}\cdot\mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ and numerical.

   (d) $\mathbf{r}\times(\mathbf{r}\times\mathbf{p}) = \mathbf{r}\,\mathbf{p}\cdot\mathbf{r} - \mathbf{p}r^2 + i\hbar\mathbf{r}$.

3. (**20 points.**) Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the following:

   (a) $\mathbf{p}\times\mathbf{L} + \mathbf{L}\times\mathbf{p} = 2i\hbar\mathbf{p}$.

   (b) $-\mathbf{L}\times\mathbf{p}\cdot\mathbf{r} = L^2$.

   (c) $\mathbf{p}\times\mathbf{L}\cdot\mathbf{p} = 2i\hbar\,p^2$.

4. (**Extra Credit.**) Using commutation relations between $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$, verify the relation

$$\mathbf{p}\times\mathbf{L}\cdot\mathbf{p} = 2i\hbar\,p^2. \tag{2}$$

Thus, verify that either of the three equalities for

$$\mathbf{M} = -\frac{1}{2}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) = -\mathbf{p}\times\mathbf{L} + i\hbar\mathbf{p} = \mathbf{L}\times\mathbf{p} - i\hbar\mathbf{p} \tag{3}$$

leads to

$$M^2 = (L^2 + \hbar^2)p^2. \tag{4}$$

Comment: This ensures that either of the following three expressions for the Axial vector

$$\mathbf{A} = \hat{\mathbf{r}} - \frac{1}{\mu Z e^2} \frac{1}{2} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) \tag{5a}$$

$$= \hat{\mathbf{r}} - \frac{1}{\mu Z e^2} \mathbf{p} \times \mathbf{L} + \frac{i\hbar}{\mu Z e^2} \mathbf{p} \tag{5b}$$

$$= \hat{\mathbf{r}} + \frac{1}{\mu Z e^2} \mathbf{L} \times \mathbf{p} - \frac{i\hbar}{\mu Z e^2} \mathbf{p} \tag{5c}$$

leads to

$$A^2 = 1 + \frac{2(L^2 + \hbar^2)H}{\mu Z^2 e^4}. \tag{6}$$

This leads to the energy levels predicted by the Bohr model, after using the Bohr quantization condition $L = n' \hbar$, where $n' = 0, 1, 2, \ldots$, and presuming that the orbit is a circle that has eccentricity $A = 0$,

$$H = -\frac{\mu Z^2 e^4}{\hbar^2} \frac{1}{2n^2}, \qquad n = 1, 2, 3, \ldots. \tag{7}$$

Show that the (classical) analysis of hydrogen atom, that does not accommodate the Heisenberg uncertainty relation, would allow the $n = 0$ energy state, that could be interpreted as a orbit with vanishing radius.