Final Exam (Spring 2018)

PHYS 510: Classical Mechanics

Date: 2018 May 11

1. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod. See Figure 1. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod.

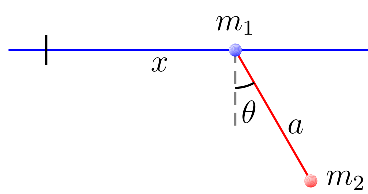

Figure 1: Problem 1.

(a) Determine the Lagrangian for the system to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 a^2 \dot{\theta}^2 + m_2 a \dot{x} \dot{\theta} \cos\theta + m_2 g a \cos\theta. \tag{1}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{x}} = m_2 \dot{x} + m_2 a \dot{\theta} \cos\theta, \tag{2a}$$

$$\frac{\partial L}{\partial x} = 0, \tag{2b}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 a^2 \dot{\theta} + m_2 a \dot{x} \cos\theta, \tag{2c}$$

$$\frac{\partial L}{\partial \theta} = -m_2 a \dot{x} \dot{\theta} \sin\theta - m_2 g a \sin\theta. \tag{2d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{x} + a\ddot{\theta} \cos\theta - a\dot{\theta}^2 \sin\theta = 0, \tag{3a}$$

$$a\ddot{\theta} + \ddot{x} \cos\theta + g \sin\theta = 0. \tag{3b}$$

Observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{x} + a\ddot{\theta} = 0, \tag{4a}$$
$$a\ddot{\theta} + \ddot{x} + g\theta = 0. \tag{4b}$$

Determine the solution to be given by

$$\theta = 0 \quad \text{and} \quad \ddot{x} = 0. \tag{5}$$

Interpret this solution.

(e) The solution $\theta = 0$ seems to be too restrictive. Will this system not allow $\theta \neq 0$? To investigate this, let us not restrict to the small angle approximation. Rewrite Eqs. (3), using Eq. (3a) in Eq. (3b), as

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = 0, \tag{6a}$$
$$\sin\theta\Big[a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta + g\Big] = 0. \tag{6b}$$

In this form we immediately observe that $\theta = 0$ is a solution. However, it is not the only solution. Towards interpretting Eqs. (6) let us identify the coordinates of the center of mass of the m_1-m_2 system,

$$(m_1 + m_2)x_{\text{cm}} = m_1 x + m_2(x + a\sin\theta), \tag{7a}$$
$$(m_1 + m_2)y_{\text{cm}} = -m_2 a\cos\theta, \tag{7b}$$

which for $m_1 = 0$ are the coordinates of the mass m_2,

$$x_{\text{cm}} = x + a\sin\theta, \tag{8a}$$
$$y_{\text{cm}} = -a\cos\theta. \tag{8b}$$

Show that

$$\dot{x}_{\text{cm}} = \dot{x} + a\dot{\theta}\cos\theta, \tag{9a}$$
$$\dot{y}_{\text{cm}} = a\dot{\theta}\sin\theta, \tag{9b}$$

and

$$\ddot{x}_{\text{cm}} = \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta, \tag{10a}$$
$$\ddot{y}_{\text{cm}} = a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta. \tag{10b}$$

Comparing Eqs. (6) and Eqs. (10) we learn that

$$\ddot{x}_{\text{cm}} = 0, \tag{11a}$$
$$\sin\theta\Big[\ddot{y}_{\text{cm}} + g\Big] = 0. \tag{11b}$$

Thus, $\ddot{y}_{\text{cm}} = -g$ is the more general solution, and $\theta = 0$ is a trivial solution.

(f) Let us analyse the system for initial conditions: $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, $\dot{x}(0) = 0$. Show that for this case $\dot{x}_{\rm cm}(0) = 0$ and

$$a(\cos\theta - \cos\theta_0) = \frac{1}{2}gt^2. \tag{12}$$

Plot θ as a function of time t. Interpret this solution.

2. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by the equation of a hyperbola

$$x^2 - c^2t^2 = x_0^2, \qquad x_0 = \frac{c^2}{\alpha}. \tag{13}$$

This is the motion of a particle 'dropped' from $x = x_0$ at $t = 0$ in region of constant (proper) acceleration.

(a) Will a photon dispatched to 'chase' this particle at $t = 0$ from $x = 0$ ever catch up with it? If yes, when and where does it catch up?

(b) Will a photon dispatched to 'chase' this particle at $t = 0$ from $0 < x < x_0$ ever catch up with it? If yes, when and where does it catch up?

(c) Will a photon dispatched to 'chase' this particle, at $t = 0$ from $x < 0$ ever catch up with it? If yes, when and where does it catch up?