Midterm Exam No. 01 (Spring 2018)

PHYS 510: Classical Mechanics

Date: 2018 Feb 27

1. (**20 points.**) Given the functional
$$F[u] = \tag{1}$$
Evaluate the functional derivative
$$\frac{\delta F[u]}{\delta u(x)}. \tag{2}$$

2. (**20 points.**) This question will be on finding the geodesic on a given surface, or to find the path of light in a medium of given refractive index.

3. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod. See Figure 3. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod.

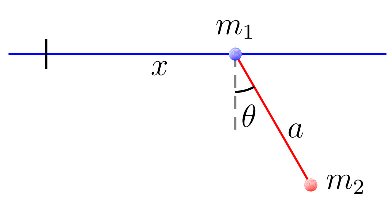

Figure 1: Problem 3.

(a) Determine the Lagrangian for the system to be
$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 a^2 \dot{\theta}^2 + m_2 a \dot{x} \dot{\theta} \cos\theta + m_2 g a \cos\theta. \tag{3}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.
$$\frac{\partial L}{\partial \dot{x}} = m_2 \dot{x} + m_2 a \dot{\theta} \cos\theta, \qquad \frac{\partial L}{\partial \dot{\theta}} = m_2 a^2 \dot{\theta} + m_2 a \dot{x} \cos\theta, \tag{4a}$$
$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial \theta} = -m_2 a \dot{x} \dot{\theta} \sin\theta - m_2 g a \sin\theta. \tag{4b}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = 0, \tag{5a}$$
$$a\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0. \tag{5b}$$

Observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) Determine the solution in the small angle approximation. Analyse it. Interpret your solution.

4. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a_2. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a frictionless hoop of radius a_1. See Figure 4. For simplification, and at loss of generality, let us chose the motion of the pendulum in the plane containing the hoop.

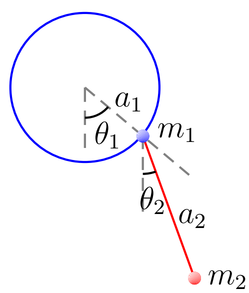

Figure 2: Problem 4.

(a) Determine the Lagrangian for the system to be

$$L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \frac{1}{2}m_2 a_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 a_2^2\dot{\theta}_2^2 + m_2 a_1 a_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$
$$+m_2 g a_1\cos\theta_1 + m_2 g a_2 \cos\theta_2. \tag{6}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_2 a_1^2\dot{\theta}_1 + m_2 a_1 a_2 \dot{\theta}_2\cos(\theta_1 - \theta_2), \tag{7a}$$
$$\frac{\partial L}{\partial \theta_1} = -m_2 a_1 a_2 \dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2 g a_1 \sin\theta_1, \tag{7b}$$
$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 a_2^2\dot{\theta}_2 + m_2 a_1 a_2 \dot{\theta}_1\cos(\theta_1 - \theta_2), \tag{7c}$$
$$\frac{\partial L}{\partial \theta_2} = m_2 a_1 a_2 \dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - m_2 g a_2 \sin\theta_2. \tag{7d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{\theta}_1 + \omega_1^2 \sin\theta_1 + \frac{1}{\beta}\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{\beta}\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{8a}$$

$$\ddot{\theta}_2 + \omega_2^2 \sin\theta_2 + \beta\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \beta\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0, \tag{8b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{9}$$

Note that β is not an independent parameter. Also, observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2\theta_1 + \frac{1}{\beta}\ddot{\theta}_2 = 0, \tag{10a}$$

$$\ddot{\theta}_2 + \omega_2^2\theta_2 + \beta\ddot{\theta}_1 = 0. \tag{10b}$$

(e) Determine the solution for the initial conditions

$$\theta_1(0) = 0, \quad \theta_2(0) = \theta_{20}, \quad \dot{\theta}_1(0) = 0, \quad \dot{\theta}_2(0) = 0. \tag{11}$$

Interpret and expound your solution.