Midterm Exam No. 01 (Spring 2018)

PHYS 510: Classical Mechanics

Date: 2018 Feb 27

1. (**20 points.**) Given the functional

$$F[u] = \int_{x_1}^{x_2} dx\, x\, u(x) \frac{du(x)}{dx}. \tag{1}$$

Assuming no variations at the end points, evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)}. \tag{2}$$

2. (**20 points.**) Find the geodesics on the surface of a circular cylinder. Identify these curves. Hint: To have a visual perception of these geodesics it helps to note that a cylinder can be mapped (or cut open) into a plane.

   (a) The distance between two points on the surface of a cylinder of radius a is characterized by the infinitesimal statement

$$ds^2 = a^2 d\phi^2 + dz^2. \tag{3}$$

   (b) The geodesic is the extremal of the functional

$$l[z] = \int_{(\phi_1, z_1)}^{(\phi_2, z_2)} ds = \int_{\phi_1}^{\phi_2} a d\phi \sqrt{1 + \left(\frac{1}{a} \frac{dz}{d\phi} \right)^2}. \tag{4}$$

   (c) Since the curve passes through the points (z_1, ϕ_1) and (z_2, ϕ_2) we have no variations on the end points. Thus, show that

$$\frac{1}{a} \frac{\delta l[z]}{\delta z(\phi)} = -\frac{d}{d\phi} \left[\frac{\dfrac{dz}{d\phi}}{\sqrt{1 + \left(\dfrac{dz}{d\phi} \right)^2}} \right]. \tag{5}$$

   (d) Using the extremum principle show that the differential equation for the geodesic is

$$\frac{1}{a} \frac{dz}{d\phi} = c, \tag{6}$$

   where c is a contant.

(e) Solve the differential equation. Identify the curves described by the solutions. Illustrate a particular curve using a diagram.

3. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod. See Figure 3. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod.

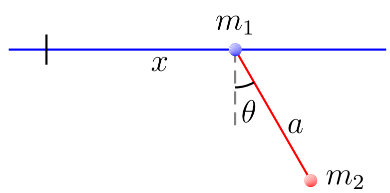

Figure 1: Problem 3.

(a) Determine the Lagrangian for the system to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2a^2\dot{\theta}^2 + m_2a\dot{x}\dot{\theta}\cos\theta + m_2ga\cos\theta. \tag{7}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{x}} = m_2\dot{x} + m_2a\dot{\theta}\cos\theta, \qquad \frac{\partial L}{\partial \dot{\theta}} = m_2a^2\dot{\theta} + m_2a\dot{x}\cos\theta, \tag{8a}$$

$$\frac{\partial L}{\partial x} = 0, \qquad \frac{\partial L}{\partial \theta} = -m_2a\dot{x}\dot{\theta}\sin\theta - m_2ga\sin\theta. \tag{8b}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = 0, \tag{9a}$$

$$a\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0. \tag{9b}$$

Observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) Determine the solution in the small angle approximation. Analyse it. Interpret your solution.

4. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a_2. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a frictionless hoop of radius a_1. See Figure 4. For simplification, and at loss of generality, let us chose the motion of the pendulum in the plane containing the hoop.

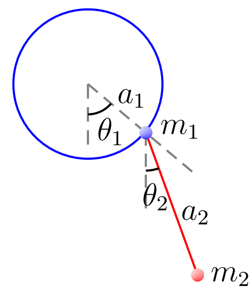

Figure 2: Problem 4.

(a) Determine the Lagrangian for the system to be

$$L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \frac{1}{2} m_2 a_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 a_2^2 \dot{\theta}_2^2 + m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 g a_1 \cos\theta_1 + m_2 g a_2 \cos\theta_2. \tag{10}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_2 a_1^2 \dot{\theta}_1 + m_2 a_1 a_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2), \tag{11a}$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g a_1 \sin\theta_1, \tag{11b}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 a_2^2 \dot{\theta}_2 + m_2 a_1 a_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2), \tag{11c}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g a_2 \sin\theta_2. \tag{11d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{\theta}_1 + \omega_1^2 \sin\theta_1 + \frac{1}{\beta} \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{\beta} \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{12a}$$

$$\ddot{\theta}_2 + \omega_2^2 \sin\theta_2 + \beta \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \beta \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0, \tag{12b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{13}$$

Note that β is not an independent parameter. Also, observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2 \theta_1 + \frac{1}{\beta} \ddot{\theta}_2 = 0, \tag{14a}$$

$$\ddot{\theta}_2 + \omega_2^2 \theta_2 + \beta \ddot{\theta}_1 = 0. \tag{14b}$$

(e) Determine the solution for the initial conditions

$$\theta_1(0) = 0, \quad \theta_2(0) = \theta_{20}, \quad \dot{\theta}_1(0) = 0, \quad \dot{\theta}_2(0) = 0. \tag{15}$$

Interpret and expound your solution.