Homework No. 06 (Spring 2018)

PHYS 510: Classical Mechanics

Due date: Tuesday, 2018 Mar 27, 4.30pm

1. (**50 points.**) (Refer Landau and Lifshitz, Problem 1 in Chapter 3.)
   A simple pendulum, consisting of a particle of mass m suspended by a string of length l in a uniform gravitational field g, is described by the Hamiltonian

$$H = \frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi. \tag{1}$$

   (a) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that

$$\frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0. \tag{2}$$

   Thus, derive

$$\frac{dt}{T_0} = \frac{1}{2\pi} \frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{3}$$

   where $T_0 = 2\pi\sqrt{l/g}$.

   (b) Determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0 \frac{2}{\pi} K\left(\sin\frac{\phi_0}{2}\right), \tag{4}$$

   where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{5}$$

   is the complete elliptic integral of the first kind.

   (c) Using the power series expansion

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} \tag{6}$$

   show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0 \left[1 + \frac{\phi_0^2}{16} + \dots\right]. \tag{7}$$

   (d) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60°$.

(e) Plot the time period T of Eq. (4) as a function of ϕ_0. What can you conclude about the time period for $\phi_0 = \pi$?

2. (**20 points.**) Starting from the Lagrangian for the Kepler problem,

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}, \tag{8}$$

derive Kepler's first law of planetary motion, which states that the orbit of a planet is a conic section. In particular, derive

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \tag{9}$$

which is the equation of a conic section in terms of the eccentricity e and the distance r_0 when the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \tag{10}$$

is minimum. We used the definitions, $L_z = \mu r^2 \dot{\phi}$,

$$r_0 = \frac{L_z^2}{\mu\alpha}, \qquad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \qquad e = \sqrt{1 + \frac{E}{U_{\text{eff}}(r_0)}}. \tag{11}$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z, which are constants of motion.