

Homework No. 07 (2018 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2018 Oct 3, 2:00 PM, in class

1. **(20 points.)** Consider an infinite chain of equidistant alternating point charges $+q$ and $-q$ on the x -axis. Calculate the electric potential at the site of a point charge due to all other charges. This is equal to the work per point charge required to assemble such a configuration. In terms of the distance a between neighbouring charges we can derive an expression for this energy to be

$$V = \frac{q}{4\pi\epsilon_0} \frac{M}{a}, \quad (1)$$

where M is a number defined as the Madelung constant for this hypothetical one-dimensional crystal. Determine M as an infinite sum, and evaluate the sum. (Madelung constants for three-dimensional crystals involve triple sums, which are typically a challenge to evaluate because of slow convergence.)

2. **(40 points.)** Consider an electric dipole at the origin, with the negative charge at the coordinate $(0, 0, -a)$ and the positive charge at $(0, 0, a)$, such that the electric dipole moment $\mathbf{p}$ points along the z -axis, $p = 2aq$.

- (a) Show that the electric potential due to the dipole at the point

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \quad r = \sqrt{x^2 + y^2 + z^2}, \quad (2)$$

is given by the expression

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+a)^2}} \right]. \quad (3)$$

- (b) For $a \ll r$ show that the potential is approximately given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{pz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}. \quad (4)$$

- (c) Consider the limit when a is made to vanish while q becomes infinite, in such a way that $2aq$ remains the finite value p . This is a point dipole. The electric potential for a point dipole is exactly described by Eq. (4). Using polar coordinates write $z = r \cos \theta$ and thus rewrite the potential of a point dipole in Eq. (4) in the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{p \cos \theta}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}. \quad (5)$$

(d) Evaluate the electric field due to a point dipole using

$$\mathbf{E} = -\nabla\phi \quad (6)$$

and express it in the following form,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]. \quad (7)$$

Draw the electric field lines of a point dipole for $\mathbf{p} = p\hat{\mathbf{z}}$.

3. **(20 points.)** (Ref. Schwinger et al., problem 4.1.)

Consider the charge density

$$\rho(\mathbf{r}) = -\mathbf{d} \cdot \nabla \delta^{(3)}(\mathbf{r}), \quad (8)$$

where $\mathbf{d}$ is constant (uniform in space).

(a) Find the total charge of the charge density by evaluating

$$\int d^3r \rho(\mathbf{r}). \quad (9)$$

Hint: Use theorem of gradient.

(b) Find the dipole moment of the charge density by evaluating

$$\int d^3r \mathbf{r} \rho(\mathbf{r}). \quad (10)$$

Hint: Integrate by parts, and use $\nabla \cdot \mathbf{r} = 3$.