Midterm Exam No. 01 (Fall 2018)

PHYS 500A: Mathematical Methods

Date: 2018 Sep 25

1. (**20 points.**) Find all z that satisfies the equation

$$e^z = e^{iz}. \tag{1}$$

2. (**20 points.**) Locate $z = \pi^i$ on the complex plane.

3. (**20 points.**) Recall that analytic functions satisfy the Cauchy-Riemann equations. That is, the real and imaginary parts of an analytic function

$$f(x + iy) = u(x, y) + iv(x, y) \tag{2}$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{3}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{4}$$

Given $f(z)$ and $g(z)$ are analytic functions in a region, then show that $f(g(z))$ satisfies the Cauchy-Riemann equations there.

4. (**20 points.**) Let

$$f(z) = \ln z, \tag{5}$$

so that

$$u(x, y) + iv(x, y) = \ln r + i\theta. \tag{6}$$

If u's are interpreted as equipotential surfaces, this represents the electrostatic configuration consisting of a line charge along the line $z = 0$. Determine the electrostatic configuration corresponding to the analytic function

$$f(z) = \ln \frac{z - a}{z + a}. \tag{7}$$

5. (**20 points.**) Show that

$$\boldsymbol{\nabla}(\hat{\mathbf{r}} \cdot \mathbf{a}) = -\frac{1}{r}\,\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{a}) \tag{8}$$

for a uniform (homogeneous in space) vector $\mathbf{a}$.

6. (**20 points.**) Cylindrical polar coordinates are defined by the transformations

$$x = \rho\cos\phi, \qquad \rho = \sqrt{x^2+y^2}, \tag{9a}$$

$$y = \rho\sin\phi, \qquad \phi = \tan^{-1}\frac{y}{x}, \tag{9b}$$

$$z = z, \qquad z = z. \tag{9c}$$

The unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ for cylindrical coordinates (ρ, ϕ) are given by

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}, \tag{10a}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}. \tag{10b}$$

The unit vector $\hat{\mathbf{k}}$ remains identical. The gradient operator in cylindrical polar coordinates is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{11}$$

A vector field in cylindrical coordinates is given by

$$\mathbf{E} = \hat{\boldsymbol{\rho}}E_\rho(\rho,\phi,z) + \hat{\boldsymbol{\phi}}E_\phi(\rho,\phi,z) + \hat{\mathbf{z}}E_z(\rho,\phi,z). \tag{12}$$

(a) Show that

$$\left(\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}\right)\cdot\mathbf{E} = \frac{\partial E_\rho}{\partial\rho}. \tag{13}$$

(b) Show that

$$\left(\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\cdot\mathbf{E} = \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{1}{\rho}E_\rho. \tag{14}$$

(c) Show that

$$\left(\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot\mathbf{E} = \frac{\partial E_z}{\partial z}. \tag{15}$$

(d) Thus, derive the expression for the divergence

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \left(\frac{1}{\rho} + \frac{\partial}{\partial\rho}\right)E_\rho + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}. \tag{16}$$

Show that the above expression is identical to the conventional expression for the divergence in cylindrical coordinates

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\right)E_\rho + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}. \tag{17}$$