Midterm Exam No. 02 (Fall 2018)

PHYS 500A: Mathematical Methods

Date: 2018 Oct 30

1. (**20 points.**) Discuss the discontinuities (branch cut) in the complex function

$$f(z) = \ln z \tag{1}$$

on the complex plane z. In particular, qualitatively discuss if the contour integrals

$$\oint_{c1} dz \, \ln z \quad \text{and} \quad \oint_{c2} dz \, \ln z \tag{2}$$

evaluate to zero using Cauchy's theorem, where the contours $c1$ and $c2$ are shown in Figure 1.

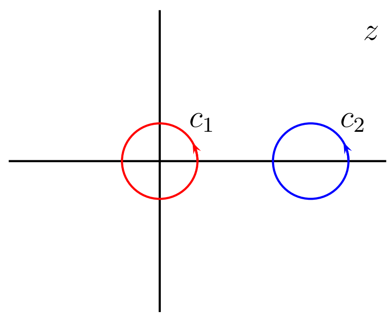

Figure 1: Contour c_1 encircles the origin while contour c_2 does not encircle the origin.

2. (**20 points.**) Evaluate the integral

$$I(a) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dx}{(x + ia)} \tag{3}$$

for $a > 0$.

3. (**20 points.**) The complex function

$$f(z) = \frac{1}{(z + 2)(z - 1)} \tag{4}$$

has the Laurent series

$$f(z) = \ldots + \frac{a_{-2}}{(z + 2)^2} + \frac{a_{-1}}{(z + 2)} + a_0 + a_1(z + 2) + a_2(z + 2)^2 + \ldots, \tag{5}$$

about $z = -2$, where

$$a_n = \frac{1}{2\pi i} \oint_c dz \frac{f(z)}{(z+2)^{n+1}}. \tag{6}$$

Choose the contour c to be a circle centered at $z = -2$ with radius less than 3 so that it does not encircle $z = 1$. Find a_n.

4. (**20 points.**) A critically damped harmonic oscillator is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\omega_0 \frac{d}{dt} + \omega_0^2\right] x(t) = 0, \tag{7}$$

where ω_0 is a characteristic frequency. Find the solution $x(t)$ for intial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Plot $x(t)$ as a function of t in the following graph where $x_0\, e^{-\omega_0 t}$ is already plotted for reference. For what t is the solution $x(t)$ a maximum?

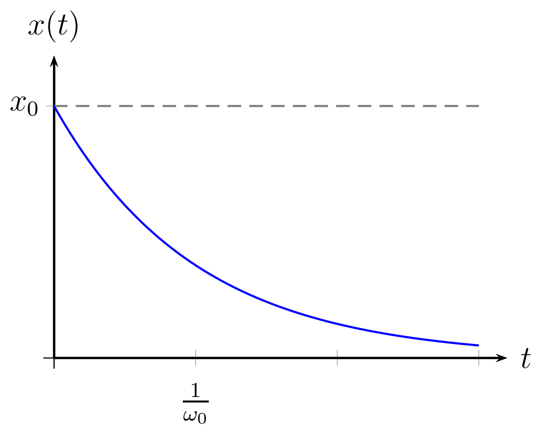

Figure 2: Critically damped harmonic oscillator.