Homework No. 02 (Fall 2018)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2018 Sep 4, 4.00pm

1. (**80 points.**)

   (a) A vector $\mathbf{v}$ in terms of the basis vectors $\mathbf{e}_i$ has the form

   $$\mathbf{v} = \mathbf{e}_i v^i, \tag{1}$$

   and in terms of another set of basis vectors $\bar{\mathbf{e}}_i$ has the form

   $$\mathbf{v} = \bar{\mathbf{e}}_i \bar{v}^i. \tag{2}$$

   If the two sets of basis vectors are related by the linear transformation

   $$\bar{\mathbf{e}}_i = \mathbf{e}_j a^j{}_i, \tag{3}$$

   then show that

   $$\bar{v}^i = b^i{}_j v^j, \tag{4}$$

   where $b = a^{-1}$.

   (b) Spherical polar coordinates are defined using the transformations

   $$x = r\sin\theta\cos\phi, \qquad r = \sqrt{x^2+y^2+z^2}, \tag{5a}$$

   $$y = r\sin\theta\sin\phi, \qquad \theta = \tan^{-1}\sqrt{\frac{x^2+y^2}{z^2}}, \tag{5b}$$

   $$z = r\cos\theta, \qquad \phi = \tan^{-1}\frac{y}{x}. \tag{5c}$$

   Let us chose

   $$\mathbf{e}_1 = \hat{\mathbf{i}}, \quad \mathbf{e}_2 = \hat{\mathbf{j}}, \quad \mathbf{e}_3 = \hat{\mathbf{k}}, \tag{6}$$

   and

   $$\bar{\mathbf{e}}_1 = \mathbf{r} = \boldsymbol{\nabla} r, \quad \bar{\mathbf{e}}_2 = \boldsymbol{\theta} = \boldsymbol{\nabla}\theta, \quad \bar{\mathbf{e}}_3 = \boldsymbol{\phi} = \boldsymbol{\nabla}\phi. \tag{7}$$

   Show that the linear transformation a connecting the two sets of basis vectors is

   $$a = \begin{bmatrix} \sin\theta\cos\phi & \dfrac{\cos\theta\cos\phi}{r} & -\dfrac{\sin\phi}{r\sin\theta} \\ \sin\theta\sin\phi & \dfrac{\cos\theta\sin\phi}{r} & \dfrac{\cos\phi}{r\sin\theta} \\ \cos\theta & -\dfrac{\sin\theta}{r} & 0 \end{bmatrix} \tag{8}$$

and

$$b = a^{-1} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{bmatrix}. \tag{9}$$

(c) The differential of a position vector in these basis set takes the form

$$d\mathbf{x} = \hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy + \hat{\mathbf{k}}dz = \mathbf{r}d\bar{x} + \boldsymbol{\theta}d\bar{y} + \boldsymbol{\phi}d\bar{z}. \tag{10}$$

Using Eq. (3) we learn that

$$\mathbf{r} = \hat{\mathbf{r}}, \quad \boldsymbol{\theta} = \frac{\hat{\boldsymbol{\theta}}}{r}, \quad \boldsymbol{\phi} = \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}, \tag{11}$$

where

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{12a}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{12b}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{12c}$$

Using Eq. (4) and replacing total differential for the sum of partial differentials we learn that

$$d\bar{x} = \sin\theta\cos\phi\,dx + \sin\theta\sin\phi\,dy + \cos\theta\,dz = dr, \tag{13a}$$
$$d\bar{y} = r\cos\theta\cos\phi\,dx + r\cos\theta\sin\phi\,dy - r\sin\theta\,dz = r^2 d\theta, \tag{13b}$$
$$d\bar{z} = -r\sin\theta\sin\phi\,dx + r\sin\theta\cos\phi\,dy = r^2\sin^2\theta d\phi. \tag{13c}$$

Thus, show that

$$d\mathbf{x} = \hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\theta}}\,rd\theta + \hat{\boldsymbol{\phi}}\,r\sin\theta d\phi. \tag{14}$$

The scale factors are read out to be

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta. \tag{15}$$

(d) The metric tensor is defined using the inner product,

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \tag{16a}$$
$$\bar{g}_{ij} = \bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j. \tag{16b}$$

Show that

$$g_{ij} = \delta_{ij} \tag{17}$$

and

$$\bar{g}_{ij} = g_{mn}a^m{}_i a^n{}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}. \tag{18}$$

Further, show that

$$d\mathbf{x}\cdot d\mathbf{x} = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{19}$$