Homework No. 03 (Fall 2018)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2018 Sep 11, 4.00pm

1. (**80 points.**) Cylindrical polar coordinates are defined by the transformations

$$x = \rho\cos\phi, \qquad \rho = \sqrt{x^2+y^2}, \tag{1a}$$

$$y = \rho\sin\phi, \qquad \phi = \tan^{-1}\frac{y}{x}, \tag{1b}$$

$$z = z, \qquad z = z. \tag{1c}$$

Cylindrical polar coordinates form a orthogonal curvilinear coordinate system. List the three family of surfaces represented by these equations. Also, list the corresponding lines of flow (normal to the these surfaces) that serve as the coordinate lines.

(a) Using the differential statement

$$dx = \frac{\partial x}{\partial \rho}d\rho + \frac{\partial x}{\partial \phi}d\phi + \frac{\partial x}{\partial z}dz \tag{2}$$

and similar relations for dy and dz show that

$$dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2. \tag{3}$$

Thus read out the scale factors for cylindrical polar coordinates as

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1. \tag{4}$$

Let

$$\mathbf{R} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Show that

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \mathbf{R} \begin{pmatrix} h_\rho d\rho \\ h_\rho d\phi \\ h_z dz \end{pmatrix}. \tag{6}$$

(b) Similarly, using the differential statement

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} \tag{7}$$

and similar relations for $\partial/\partial y$ and $\partial/\partial z$ show that

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \frac{1}{h_\rho}\frac{\partial}{\partial \rho} \\ \frac{1}{h_\phi}\frac{\partial}{\partial \phi} \\ \frac{1}{h_z}\frac{\partial}{\partial z} \end{pmatrix}. \tag{8}$$

(c) Using the construction of the unit vector

$$\hat{\boldsymbol{\rho}} = \frac{\boldsymbol{\nabla}\rho}{|\boldsymbol{\nabla}\rho|} \tag{9}$$

and similar constructions for $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{z}}$ show that

$$\begin{pmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \hat{\boldsymbol{\rho}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix}. \tag{10}$$

(d) Starting from the differential statement in rectangular coordinates

$$d\mathbf{r} = dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}} \tag{11}$$

derive the corresponding differential statement in cylindrical polar coordinates

$$d\mathbf{r} = h_\rho d\rho\, \hat{\boldsymbol{\rho}} + h_\phi d\phi\, \hat{\boldsymbol{\phi}} + h_z dz\, \hat{\mathbf{z}}. \tag{12}$$

(e) Using the expression for the gradient operator in rectangular coordinates

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{13}$$

derive the expression for the gradient operator in cylindrical polar coordinates

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{1}{h_\rho}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{h_\phi}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{1}{h_z}\frac{\partial}{\partial z}. \tag{14}$$

(f) Let a function $f(x, y, z)$ in cylindrical polar coordinates be $\bar{f}(\rho, \phi, z)$. That is, $f(x, y, z) = \bar{f}(\rho, \phi, z)$. Show that

$$df = d\mathbf{r} \cdot \boldsymbol{\nabla} f = \left[dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z} \right] f(x, y, z). \tag{15}$$

Show that

$$df = \left[d\rho\frac{\partial}{\partial \rho} + d\phi\frac{\partial}{\partial \phi} + dz\frac{\partial}{\partial z} \right] \bar{f}(\rho, \phi, z) = d\bar{f}. \tag{16}$$

This is the definition of a scalar field.

(g) Consider the vector field in rectangular coordinates

$$\mathbf{E} = \hat{\mathbf{i}}E_x(x, y, z) + \hat{\mathbf{j}}E_y(x, y, z) + \hat{\mathbf{k}}E_z(x, y, z). \tag{17}$$

A vector field, by definition, in cylindrical coordinates is given by

$$\begin{pmatrix} E_x(x, y, z) \\ E_y(x, y, z) \\ E_z(x, y, z) \end{pmatrix} = \mathbf{R} \begin{pmatrix} E_\rho(\rho, \phi, z) \\ E_\phi(\rho, \phi, z) \\ E_z(\rho, \phi, z) \end{pmatrix}. \tag{18}$$

Thus, show that

$$\mathbf{E} = \hat{\boldsymbol{\rho}}E_\rho(\rho, \phi, z) + \hat{\boldsymbol{\phi}}E_\phi(\rho, \phi, z) + \hat{\mathbf{z}}E_z(\rho, \phi, z). \tag{19}$$

(h) Derive the following nine derivatives

$$\begin{pmatrix} \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \rho} & \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \rho} & \frac{\partial \hat{\mathbf{z}}}{\partial \rho} \\ \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi} & \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} & \frac{\partial \hat{\mathbf{z}}}{\partial \phi} \\ \frac{\partial \hat{\boldsymbol{\rho}}}{\partial z} & \frac{\partial \hat{\boldsymbol{\phi}}}{\partial z} & \frac{\partial \hat{\mathbf{z}}}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \hat{\boldsymbol{\phi}} & -\hat{\boldsymbol{\rho}} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

(i) Show that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{h_\rho h_\phi h_z} \left[\frac{\partial}{\partial \rho}(h_\phi h_z E_\rho) + \frac{\partial}{\partial \phi}(h_z h_\rho E_\phi) + \frac{\partial}{\partial z}(h_\rho h_\phi E_z) \right]. \tag{21}$$

(j) Show that

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{1}{h_\rho h_\phi h_z} \begin{vmatrix} h_\rho \hat{\boldsymbol{\rho}} & h_\phi \hat{\boldsymbol{\phi}} & h_z \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ h_\rho E_\rho & h_\phi E_\phi & h_z E_z \end{vmatrix}. \tag{22}$$

(k) Show that

$$\nabla^2 = \frac{1}{h_\rho h_\phi h_z} \left[\frac{\partial}{\partial \rho} \frac{h_\phi h_z}{h_\rho} \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi} \frac{h_z h_\rho}{h_\phi} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial z} \frac{h_\rho h_\phi}{h_z} \frac{\partial}{\partial z} \right]. \tag{23}$$