Homework No. 04 (Fall 2018)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2018 Sep 18, 4.00pm

1. (**30 points.**) The close connection between the geometry of a complex number

$$z = x + iy \tag{1}$$

and a two-dimensional vector

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \tag{2}$$

is intriguing. They have the same rules for addition and subtraction, but differ in their rules for multiplication. Show that

$$z_1^* z_2 = (\mathbf{r}_1 \cdot \mathbf{r}_2) + i(\mathbf{r}_1 \times \mathbf{r}_2) \cdot \hat{\mathbf{k}}. \tag{3}$$

In the quest for a number system that corresponds to a three dimensional vector, Hamilton in 1843 invented the quaternions. A quaternion P can be expressed in terms of Pauli matrices as

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}. \tag{4}$$

Recall that the Pauli matrices are completely characterized by the identity

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b}) + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{5}$$

(a) Show that the (Hamilton) product of two quaternions,

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}, \tag{6a}$$

$$Q = b_0 - i\mathbf{b} \cdot \boldsymbol{\sigma}, \tag{6b}$$

is given by

$$PQ = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) - i(a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{7}$$

(b) Verify that the Hamilton product is non-commutative. Determine $\big[P, Q\big]$.
Solution:

$$\big[P, Q\big] = -2i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{8}$$

2. (**20 points.**) Let

$$u(x, y) = \frac{x^3 - y^3}{x^2 + y^2}. \tag{9}$$

Then evaluate the following:

$$\lim_{x \to 0} \lim_{y \to 0} \frac{\partial u}{\partial x}, \tag{10a}$$

$$\lim_{y \to 0} \lim_{x \to 0} \frac{\partial u}{\partial x}. \tag{10b}$$

Are they equal?