Homework No. 05 (Fall 2018)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2018 Sep 25, 4.00pm

1. (**20 points.**) Let
$$f(z) = z^3, \tag{1}$$
so that
$$u(x,y) + iv(x,y) = r^3(\cos 3\theta + i\sin 3\theta). \tag{2}$$

   (a) Verify that this function satisifes the Cauchy-Riemann conditions.

   (b) Show that u and v are harmonic functions. That is, they satisfy the Laplacian. Further, show that
$$(\boldsymbol{\nabla} u)\cdot(\boldsymbol{\nabla} v) = 0. \tag{3}$$
   Thus, the curves represented by u and v are orthogonal at every point.

   (c) Since u is a harmonic function it represents equipotential curves. Plot the equipotentials
$$r = \left[\frac{u}{\cos 3\theta}\right]^{\frac{1}{3}} \tag{4}$$
   for $u = -10, -1, -0.1, 0, 0.1, 1, 10$. In Mathematica this can be achieved using the command

```
PolarPlot[{r[-10],...,r[10]},{theta,0,2 Pi}],
```

   where `r[u]` is a function that needs to be defined ahead.

   (d) Determine the electric field associated to these equipotentials using
$$\mathbf{E} = -\boldsymbol{\nabla} u. \tag{5}$$
   This is easily achieved using
$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \tag{6}$$
   and similarly for derivatives with respect to y. Recall
$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}. \tag{7}$$
   Show that
$$\mathbf{E} = -\hat{\mathbf{i}}\, 3r^2\cos 2\theta + \hat{\mathbf{j}}\, 3r^2 \sin 2\theta. \tag{8}$$

(e) The curves representing the field lines are obtained by requiring the tangent lines for these curves to have the same slope as the electric field,

$$\frac{dx}{E_x} = \frac{dy}{E_y}. \tag{9}$$

Rewrite this equation as

$$E_y dx - E_x dy = 0. \tag{10}$$

Comparing this equation with

$$\frac{\partial s}{\partial x}\, dx + \frac{\partial s}{\partial y}\, dy = 0 \tag{11}$$

identify the equations satisfied by the curves $s(x, y)$, representing the field lines associated to the equipotentials u, as

$$\frac{\partial s}{\partial x} = 6xy, \qquad \frac{\partial s}{\partial y} = 3(x^2 - y^2). \tag{12}$$

Solve these equations to determine the equations for the field lines to be

$$s(x, y) = 3x^2 y - y^3 = r^3 \sin 3\theta \tag{13}$$

up to a constant. The field lines s are indeed v. Plot the field lines

$$r = \left[\frac{v}{\sin 3\theta}\right]^{\frac{1}{3}} \tag{14}$$

for $v = -10, -1, -0.1, 0, 0.1, 1, 10$.

(f) Plot the equipotentials in red and field lines in blue in the same plot. Here is a simple code for it in Mathematica

```
n = 3;
f[u_] = (u/Cos[n t])^(1/n);
g[u_] = (u/Sin[n t])^(1/n);
PolarPlot[
 {f[-10], f[-1], f[-0.1], f[0], f[0.1], f[1], f[10],
  g[-10], g[-1], g[-0.1], g[0], g[0.1], g[1], g[10]},
 {t, -Pi, Pi},
 PlotStyle -> {Red, Red, Red, Red, Red, Red, Red,
               Blue, Blue, Blue, Blue, Blue, Blue, Blue},
 PlotRange -> {-4, 4}]
```

which generates the plots in Fig. 1.

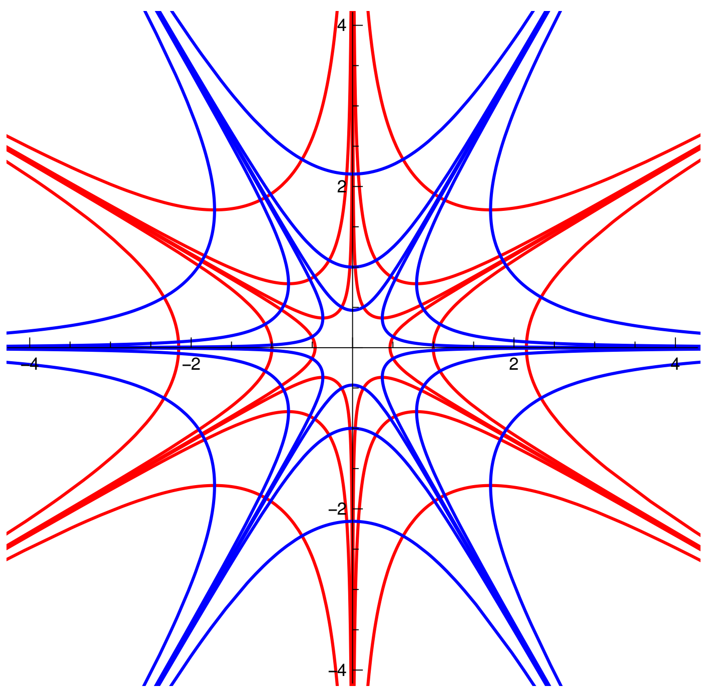

Figure 1: Equipotentials and field lines represented by the analytic function $f(z) = z^3$.

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