Homework No. 06 (Fall 2018)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2018 Oct 16, 4.00pm

1. (**90 points.**) Consider the integral

$$I(v,w) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{w^2 + vw\cos\theta}{v^2 + w^2 + 2vw\cos\theta}, \tag{1}$$

where v and w are complex.

(a) Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{2}$$

and express the integral as a contour integral,

$$I(v,w) = \frac{1}{2\pi i}\frac{1}{2}\oint_c \frac{dz}{z} \frac{z^2 + 2\frac{w}{v}z + 1}{\left(z + \frac{v}{w}\right)\left(z + \frac{w}{v}\right)}, \tag{3}$$

where the contour c is along the unit circle going counterclockwise. Locate the three poles, $z = 0$, $z = -v/w$, and $z = -w/v$.

(b) Evaluate the residues and show that

$$I(v,w) = \begin{cases} 1, & \text{if } |v| < |w|, \\ 0, & \text{if } |w| < |v|. \end{cases} \tag{4}$$

Observe that for $v = w$, (which is more restrictive than $|v| = |w|$,) we have

$$I(v,w) = \frac{1}{2}. \tag{5}$$

(c) Let us seek the partial fraction decomposition

$$\frac{z^2 + 2\frac{w}{v}z + 1}{z\left(z + \frac{v}{w}\right)\left(z + \frac{w}{v}\right)} = \frac{a}{z} + \frac{b}{\left(z + \frac{v}{w}\right)} + \frac{c}{\left(z + \frac{w}{v}\right)}. \tag{6}$$

Show that $a = 1$, $b = 1$, and $c = -1$. Thus, express the integral in the form

$$I(v,w) = \frac{1}{2}\left[\frac{1}{2\pi i}\oint_c \frac{dz}{z} + \frac{1}{2\pi i}\oint_c \frac{dz}{\left(z + \frac{v}{w}\right)} - \frac{1}{2\pi i}\oint_c \frac{dz}{\left(z + \frac{w}{v}\right)}\right] \tag{7}$$

(d) Show that

$$\frac{1}{2\pi i}\oint_c \frac{dz}{z} = 1. \tag{8}$$

Evaluate the integrals

$$\frac{1}{2\pi i}\oint_c \frac{dz}{\left(z+\frac{v}{w}\right)} = \theta\left(1-\frac{|v|}{|w|}\right), \tag{9a}$$

$$\frac{1}{2\pi i}\oint_c \frac{dz}{\left(z+\frac{w}{v}\right)} = \theta\left(1-\frac{|w|}{|v|}\right), \tag{9b}$$

where $\theta(x)$ is the Heaviside step function. Thus, derive the relation

$$I(v,w) = \frac{1}{2}\left[1+\theta\left(1-\frac{|v|}{|w|}\right)-\theta\left(1-\frac{|w|}{|v|}\right)\right] \tag{10}$$

and verify Eq. (4).

(e) What electrostatic configuration in two dimensions represents the complex function in Eq. (6).