Homework No. 08 (Fall 2018)

PHYS 500A: Mathematical Methods

Due date: Tuesday, 2018 Oct 30, 4.00pm

1. (**80 points.**) A damped harmonic oscillator, constituting of a body of mass m and a spring of spring constant k, is described by

$$ma = -kx - bv, \tag{1}$$

where x is position, $v = dx/dt$ is velocity, $a = dv/dt$ is acceleration, and b is the damping coefficient. Thus, we have the differential equation

$$\left[\frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2\right] x(t) = 0 \tag{2}$$

with intial conditions

$$x(0) = x_0, \tag{3a}$$
$$\dot{x}(0) = v_0, \tag{3b}$$

where

$$\omega_0^2 = \frac{k}{m}, \qquad 2\gamma = \frac{b}{m}. \tag{4}$$

(a) $\gamma = 0$: In the absence of damping show that the solution is

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t. \tag{5}$$

(b) $\gamma < \omega_0$: Underdamped harmonic oscillator.

$$x(t) = e^{-\gamma t} \left[x_0 \cos \sqrt{\omega_0^2 - \gamma^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2} t\right]. \tag{6}$$

(c) $\gamma = \omega_0$: Critically damped harmonic oscillator.

$$x(t) = e^{-\omega_0 t} \left[x_0 + (v_0 + \omega_0 x_0) t\right]. \tag{7}$$

(d) $\gamma > \omega_0$: Overdamped harmonic oscillator.

$$x(t) = e^{-\gamma t} \left[x_0 \cosh \sqrt{\gamma^2 - \omega_0^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh \sqrt{\gamma^2 - \omega_0^2} t\right]. \tag{8}$$

(e) Set $\omega_0 = 1$, which is equivalent to the substitution $\omega_0 t = \tau$, and sets the scale for the time t. That is, time is measured in units of $T = 2\pi/\omega_0$. The system is then completely characterized by the parameter γ/ω_0 and the initial conditions x_0 and v_0. Plot the solutions for the initial conditions $x_0 = 0$ and $v_0 = 1$.