# Homework No. 10 (Fall 2018)

## PHYS 500A: Mathematical Methods

Due date: Thursday, 2018 Nov 29, 4.00pm

1. (**40 points.**) Recollect Legendre polynomials

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{1}$$

In particular

$$P_0(x) = 1, \tag{2a}$$
$$P_1(x) = x, \tag{2b}$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{2c}$$

Consider a charged spherical shell of radius $a$ consisting of a charge distribution in the polar angle alone,

$$\rho(\mathbf{r}') = \sigma(\theta')\,\delta(r'-a). \tag{3}$$

The electric potential *on the z-axis*, $\theta = 0$ and $\phi = 0$, is then given by

$$\begin{aligned} \phi(r,0,0) &= \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{2\pi a^2}{4\pi\varepsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}}, \end{aligned} \tag{4}$$

after evaluating the $r'$ and $\phi'$ integral.

(a) Consider a uniform charge distribution on the shell,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_0(\cos\theta). \tag{5}$$

Evaluate the integral in Eq. (4) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>}, \tag{6}$$

where $r_< = \text{Min}(a,r)$ and $r_> = \text{Max}(a,r)$.
Note: This was done in class. Nevertheless, present the relevant steps.

(b) Next, consider a (pure dipole, $2 \times 1$-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_1(\cos\theta). \tag{7}$$

Evaluate the integral in Eq. (4) to show that

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{3} \frac{1}{r_>} \left( \frac{r_<}{r_>} \right). \tag{8}$$

Note: This was done in class. Nevertheless, present the relevant steps.

(c) Next, consider a (pure quadrapole, $2 \times 2$-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_2(\cos\theta). \tag{9}$$

Evaluate the integral in Eq. (4) to show that

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{5} \frac{1}{r_>} \left( \frac{r_<}{r_>} \right)^2. \tag{10}$$

(d) For a (pure $2l$-pole) charge distribution

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_l(\cos\theta) \tag{11}$$

the integral in Eq. (4) leads to

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{(2l+1)} \frac{1}{r_>} \left( \frac{r_<}{r_>} \right)^l. \tag{12}$$