Homework No. 11 (Fall 2018)

PHYS 500A: Mathematical Methods

Due date: Thursday, 2018 Dec 6, 4.00pm

1. (**40 points.**) Generate 3D plots of surface spherical harmonics $Y_{lm}(\theta,\phi)$ as a function of θ and ϕ. In particular,

   (a) Plot $\text{Re}\big[Y_{73}(\theta,\phi)\big]$.

   (b) Plot $\text{Im}\big[Y_{73}(\theta,\phi)\big]$.

   (c) Plot $\text{Abs}\big[Y_{73}(\theta,\phi)\big]$.

   (d) Plot your favourite spherical harmonic, that is, choose a l and m, and Re or Im or Abs.

   Hint: In Mathematica these plots are generated using the following commands:
   ```
   SphericalPlot3D[Re[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   SphericalPlot3D[Im[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   SphericalPlot3D[Abs[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   ```
   Refer to diagrams in Wikipedia article on 'spherical harmonics' to see some visual representations of these functions.

2. (**20 points.**) Verify that the right hand side of

$$(-\mathbf{a}\cdot\boldsymbol{\nabla})\frac{1}{r} = \frac{\mathbf{a}\cdot\mathbf{r}}{r^3} \tag{1}$$

   is a solution to Laplace's equation for $\mathbf{r}\neq 0$. Further, verify the relation

$$(\mathbf{a}_1\cdot\boldsymbol{\nabla})(\mathbf{a}_2\cdot\boldsymbol{\nabla})\frac{1}{r} = \frac{1}{r^5}\Big[3(\mathbf{a}_1\cdot\mathbf{r})(\mathbf{a}_2\cdot\mathbf{r}) - (\mathbf{a}_1\cdot\mathbf{a}_2)r^2\Big], \tag{2}$$

   which is also a solution to Laplace's equation for $\mathbf{r}\neq 0$, but need not be verified here.

3. (**10 points.**) The generating function for the spherical harmonics, $Y_{lm}(\theta,\phi)$, is

$$\frac{1}{l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\,Y_{lm}(\theta,\phi)\,\psi_{lm}, \tag{3}$$

   where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{4}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \tag{5}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{6}$$

and

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{7}$$

Show that

$$\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right) \tag{8}$$

is unchanged by the substitution: $y_+ \leftrightarrow y_-$, $\theta \to -\theta$, $\phi \to -\phi$. Thus, show that

$$Y_{lm}(\theta,\phi) = Y_{l,-m}(-\theta,-\phi). \tag{9}$$

4. (**30 points.**) Write down the explicit forms of the spherical harmonics $Y_{lm}(\theta,\phi)$ for $l = 0, 1, 2$, by completing the $l - m$ differentiations in Eq. (7). Use the result in Eq. (9) to reduce the work by about half.

5. (**50 points.**) Legendre polynomials of order l is given by (for $|t| < 1$)

$$P_l(t) = \left(\frac{d}{dt}\right)^l \frac{(t^2-1)^l}{2^l l!}. \tag{10}$$

   (a) Write down the explicit forms of the Legendre polynomials $P_l(t)$ for $l = 0, 1, 2, 3$, by completing the l differentiations in Eq. (10).

   (b) Show that the spherical harmonics for $m = 0$ involves the Legendre polynomials,

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta). \tag{11}$$