Final Exam (Fall 2018)

PHYS 520A: Electromagnetic Theory I

Date: 2018 Dec 12

1. (**30 points.**) A simple model for susceptibility is

$$\chi(\omega) = \frac{\omega_1}{\omega_0 - \omega} + i\,\pi\omega_1\delta(\omega - \omega_0), \tag{1}$$

where ω_0 and ω_1 represent physical parameters of a material. Note that

$$[\mathrm{Re}\chi(\omega)] = \frac{\omega_1}{\omega_0 - \omega} \qquad \text{and} \qquad [\mathrm{Im}\chi(\omega)] = \pi\omega_1\delta(\omega - \omega_0). \tag{2}$$

(a) Qualitatively plot $[\mathrm{Re}\chi(\omega)]$ and $[\mathrm{Im}\chi(\omega)]$ with respect to ω.

(b) Evaluate the right hand side of the Kramers-Kronig relation

$$[\mathrm{Re}\chi(\omega)] = \lim_{\delta\to 0+} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} [\mathrm{Im}\chi(\omega')]\, 2\mathrm{Re}\left\{ \frac{1}{\omega' - (\omega + i\delta)} \right\} \tag{3}$$

for this simple model.

2. (**30 points.**) Consider a uniformly polarized sphere described by

$$\mathbf{P}(\mathbf{r}) = \alpha\,\mathbf{r}\,\theta(R - r). \tag{4}$$

(a) Calculate $-\boldsymbol{\nabla}\cdot\mathbf{P}$. Thus, find the effective charge density to be

$$\rho_{\mathrm{eff}} = -3\alpha\theta(R - r) + \alpha R\delta(r - R). \tag{5}$$

(b) Find the enclosed charge inside a sphere of radius r using

$$Q_{\mathrm{en}} = \int d^3r\, \rho_{\mathrm{eff}}(\mathbf{r}) \tag{6}$$

for $r < R$ and $r > R$.

(c) Use Gauss's law to find the electric field for $r < R$ and $r > R$.

3. (**40 points.**) The expression for the electric potential due to a point charge placed in between two perfectly conducting semi-infinite slabs described by

$$\varepsilon(z) = \begin{cases} \infty, & z < 0, \\ \varepsilon_0, & 0 < z < a, \\ \infty, & a < z, \end{cases} \tag{7}$$

is given in terms of the reduced Green's function that satisfies the differential equation $(0 < \{z, z'\} < a)$

$$\left[-\frac{\partial^2}{\partial z^2} + k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{8}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and $z = a$.

(a) Construct the reduced Green's function in the form

$$g(z, z') = \begin{cases} A \sinh kz + B \cosh kz, & 0 < z < z' < a, \\ C \sinh kz + D \cosh kz, & 0 < z' < z < a, \end{cases} \tag{9}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$g(0, z') = 0, \tag{10a}$$

$$g(a, z') = 0, \tag{10b}$$

$$g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{10c}$$

$$\varepsilon_0 \partial_z g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{10d}$$

Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{11}$$

(b) Take the limit $ka \to \infty$ in your solution above, (which corresponds to moving the slab at $z = a$ to infinity,) to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka \to \infty} g(z, z') = \frac{1}{\varepsilon_0} \frac{1}{2k} e^{-k|z-z'|} - \frac{1}{\varepsilon_0} \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{12}$$

This should serve as a check for your solution to the reduced Green's function.