Midterm Exam No. 01 (Fall 2018)

PHYS 520A: Electromagnetic Theory I

Date: 2018 Sep 28

1. (**20 points.**) (Based on Schwinger et al., problem 7, chapter 1.)
   A charge q moves in the vacuum under the influence of uniform fields $\mathbf{E}$ and $\mathbf{B}$. The force on this charge is given by the Lorentz force
   $$\mathbf{F} = q\big[\mathbf{E} + \mathbf{v} \times \mathbf{B}\big]. \tag{1}$$
   Assume that $\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{v} \cdot \mathbf{B} = 0$.
   (a) For what (magnitude and direction of) velocity does the charge move without acceleration, that is, $\mathbf{F} = 0$?
   (b) What is the speed when $\sqrt{\varepsilon_0}|\mathbf{E}| = |\mathbf{B}|/\sqrt{\mu_0}$?
   (Remember, speed of light c in Maxwell's equations is identified using $\varepsilon_0\mu_0 = 1/c^2$.)
   (c) Give a realization of the physical situation in item (1b) and comment on it intuitively. (This part of the question might not be graded.)

2. (**20 points.**) A point dipole $\mathbf{p}$, stationary at position $\mathbf{r}_0$, is described by the charge density
   $$\rho(\mathbf{r}, t) = -\mathbf{p} \cdot \boldsymbol{\nabla}\delta^{(3)}(\mathbf{r} - \mathbf{r}_0). \tag{2}$$
   Determine the force on the point dipole in an electric field $\mathbf{E}(\mathbf{r}, t)$.
   Hint: Force on the dipole is given by the integral of the Lorentz force density
   $$\mathbf{f}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\mathbf{E}(\mathbf{r}, t) + \mathbf{j}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t). \tag{3}$$
   Evaluate the integral using the properties of δ-function.

3. (**20 points.**) A plane wave is described by electric and magnetic fields of the form
   $$\mathbf{E} = \mathbf{e}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{4}$$
   $$\mathbf{B} = \mathbf{b}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{5}$$
   where $\mathbf{e}_0$ and $\mathbf{b}_0$ are constants. Assume no charges or currents.
   (a) Using Maxwell's equations evaluate
   $$\mathbf{k} \cdot \mathbf{E}, \quad \mathbf{k} \cdot \mathbf{B}, \quad \mathbf{k} \times \mathbf{E}, \quad \text{and} \quad \mathbf{k} \times \mathbf{B}. \tag{6}$$

(b) Further, using Eqs. (6), derive the relations

$$ck = \omega, \quad \hat{\mathbf{k}} \times \mathbf{e}_0 = c\mathbf{b}_0, \quad c\hat{\mathbf{k}} \times \mathbf{b}_0 = -\mathbf{e}_0, \quad \text{and} \quad cb_0 = e_0, \tag{7}$$

where $\varepsilon_0\mu_0 = 1/c^2$.

(c) Evaluate the energy density

$$U = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2 \tag{8}$$

and the momentum density

$$\mathbf{G} = \varepsilon_0 \mathbf{E} \times \mathbf{B}. \tag{9}$$

Then, determine the ratio U/G.

4. (**60 points.**) Let us consider the static configuration of a point electric charge q_e at position $\mathbf{r}_e$ and a point magnetic charge q_m at position $\mathbf{r}_m$. Let $\mathbf{r}_e - \mathbf{r}_m = \mathbf{a}$. For convenience you can choose the magnetic charge at the origin and the electric charge on the z axis.

(a) Using Gauss's law show that the electric field for a point electric charge is given by

$$\mathbf{E} = \frac{q_e}{4\pi\varepsilon_0} \frac{(\mathbf{r} - \mathbf{r}_e)}{|\mathbf{r} - \mathbf{r}_e|^3} = -\boldsymbol{\nabla}\phi_e, \qquad \phi_e = \frac{q_e}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_e|} \tag{10}$$

and the magnetic field for a point magnetic charge is

$$\mathbf{H} = \frac{q_m}{4\pi\mu_0} \frac{(\mathbf{r} - \mathbf{r}_m)}{|\mathbf{r} - \mathbf{r}_m|^3} = -\boldsymbol{\nabla}\phi_m, \qquad \phi_m = \frac{q_m}{4\pi\mu_0} \frac{1}{|\mathbf{r} - \mathbf{r}_m|}. \tag{11}$$

(b) Show that the electromagnetic momentum density

$$\mathbf{G} = \mathbf{D} \times \mathbf{B} \tag{12}$$

for this configuration is

$$\mathbf{G} = \varepsilon_0\mu_0 (\boldsymbol{\nabla}\phi_e) \times (\boldsymbol{\nabla}\phi_m). \tag{13}$$

Show that $\boldsymbol{\nabla} \cdot \mathbf{G} = 0$. What is the interpretation? Thus, infer that $\mathbf{G}$ can be expressed as a curl.

(c) Show that the angular momentum density

$$\boldsymbol{l} = \mathbf{r} \times \mathbf{G} \tag{14}$$

for this configuration is

$$\boldsymbol{l} = (\mathbf{r} \cdot \mathbf{B})\mathbf{D} - (\mathbf{r} \cdot \mathbf{D})\mathbf{B}. \tag{15}$$

(d) The angular momentum is

$$\mathbf{L} = \int d^3r\, \boldsymbol{l}, \tag{16}$$

where the integration is over all space. Show that

$$\int d^3r\, (\mathbf{r}\cdot\mathbf{B})\mathbf{D} = \varepsilon_0\mu_0 \int d^3r \Big[3\mathbf{E} + (\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{E})\Big]\phi_m, \tag{17a}$$

$$\int d^3r\, (\mathbf{r}\cdot\mathbf{B})\mathbf{D} = \varepsilon_0\mu_0 \int d^3r \Big[\mathbf{E} + (\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{E})\Big]\phi_m. \tag{17b}$$

Thus, deduce the angular momentum for this configuration to be

$$\mathbf{L} = \frac{q_e q_m}{2\pi}\frac{1}{4\pi}\int d^3r\, \frac{(\mathbf{r}-\mathbf{r}_e)}{|\mathbf{r}-\mathbf{r}_e|^3}\frac{1}{|\mathbf{r}-\mathbf{r}_m|}. \tag{18}$$

(e) Recall that the electric field due to charge distribution $\rho(\mathbf{r}')$ at the point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\, \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}'). \tag{19}$$

Thus, the integrals leading to the angular momentum in Eq. (18) can be performed by evaluating the electric field due to a charge density that is inversely proportional to distance. Using Gauss's law find the electric field at the point $\mathbf{r}_e$ due to a charge density

$$\rho(\mathbf{r}) = \frac{\sigma}{|\mathbf{r}-\mathbf{r}_m|} \tag{20}$$

to be

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\,\hat{\mathbf{a}}, \tag{21}$$

where $\mathbf{a} = \mathbf{r}_e - \mathbf{r}_m$.

(f) Evaluate the angular momentum in Eq. (18) to be

$$\mathbf{L} = -\frac{q_e q_m}{4\pi}\,\hat{\mathbf{a}}. \tag{22}$$

(g) Using the Bohr quantization condition

$$L = n\hbar \tag{23}$$

derive the charge quantization condition of Dirac,

$$\frac{q_e q_m}{4\pi} = n\hbar. \tag{24}$$