Midterm Exam No. 02 (Fall 2018)

PHYS 520A: Electromagnetic Theory I

Date: 2018 Nov 9

1. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{x} dx'\, \delta(x'). \tag{1}$$

2. (**40 points.**) We showed in class that the electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{d}\cdot\hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{2}$$

The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by

$$U = -\mathbf{d}\cdot\mathbf{E}. \tag{3}$$

Further, the force between the two dipoles is given by

$$\mathbf{F} = -\boldsymbol{\nabla} U. \tag{4}$$

Use these expressions to derive

(a) the interaction energy between two point dipoles separated by distance $\mathbf{r}$ to be

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[\mathbf{d}_1\cdot\mathbf{d}_2 - 3\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\big]. \tag{5}$$

(b) the force between the two dipoles to be

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{3}{r^4}\big[(\mathbf{d}_1\cdot\mathbf{d}_2)\,\hat{\mathbf{r}} + (\mathbf{d}_1\cdot\hat{\mathbf{r}})\,\mathbf{d}_2 + (\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{d}_1 - 5\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\big]. \tag{6}$$

(c) Are the forces central? That is, is the force in the direction of $\mathbf{r}$?

(d) Are the forces on the dipole equal in magnitude and opposite in direction? That is, do they satisfy Newton's third law?

3. (**40 points.**) Consider a solid sphere of radius R with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0\,\theta(R - r), \tag{7}$$

where $\mathbf{P}_0$ is a uniform vector, $\theta(x)$ is the Heaviside step function, and $r^2 = x^2 + y^2 + z^2$. We shall find the electric potential and the associated electric field inside and outside the sphere.

(a) Show that the effective charge density due to the polarization is

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} = (\mathbf{P}_0\cdot\hat{\mathbf{r}})\,\delta(r-R). \tag{8}$$

(b) Beginning from

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho_{\text{eff}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{9}$$

show that

$$\phi(\mathbf{r}) = \frac{R^2}{4\pi\varepsilon_0}\int d\Omega\,\frac{(\mathbf{P}_0\cdot\hat{\mathbf{R}})}{|\mathbf{r}-\mathbf{R}|}. \tag{10}$$

Here bfr is the observation point and the integration spans the surface of the sphere, $d\Omega = \sin\theta d\theta d\phi$ and $\mathbf{R}$ is the radius vector. More explicitly we have

$$\phi(\mathbf{r}) = \frac{R^2}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi'\int_0^{\pi}\sin\theta' d\theta'\,\frac{\mathbf{P}_0\cdot(\sin\theta'\cos\phi'\hat{\mathbf{i}}+\sin\theta'\sin\phi'\hat{\mathbf{j}}+\cos\theta'\hat{\mathbf{k}})}{\sqrt{r^2+R^2-2rR\cos\gamma}}, \tag{11}$$

where γ is the angle between the vectors $\mathbf{r}$ and $\mathbf{R}$ and is given by

$$\cos\gamma = \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'. \tag{12}$$

(c) Out of the three vectors $\mathbf{P}_0$, $\mathbf{r}$, and $\mathbf{R}$, choose the z axis to be along $\mathbf{r}$. This renders

$$\gamma = \theta' \tag{13}$$

and allows for the integration to be completed using elementary substitutions. Legendre introduced the polynomials named after him primarily to evaluate these integrals without this specific choice.

Complete the ϕ' integral to yield

$$\phi(\mathbf{r}) = \frac{(\mathbf{P}_0\cdot\hat{\mathbf{k}})}{4\pi\varepsilon_0}2\pi R^2\int_0^{\pi}\sin\theta' d\theta'\,\frac{\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}. \tag{14}$$

(d) Evaluate the θ' integral and show that

$$\int_0^{\pi}\sin\theta' d\theta'\,\frac{\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}} = \begin{cases}\dfrac{2}{3}\dfrac{r}{R^2}, & r<R,\\[2mm] \dfrac{2}{3}\dfrac{R}{r^2}, & R<r.\end{cases} \tag{15}$$

(e) Thus, find the electric potential. Also, release the choice of the z axis along $\mathbf{r}$ by replacing $\mathbf{k}$ with $\mathbf{r}$. Show that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{(\mathbf{P}_0\cdot\hat{\mathbf{r}})}{r^2}\begin{cases}\dfrac{4\pi}{3}r^3, & r<R,\\[2mm] \dfrac{4\pi}{3}R^3, & R<r.\end{cases} \tag{16}$$

Compare this to the electric potential of a point dipole. Are they identical?

(f) Determine the electric field using

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{17}$$

Show that

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \begin{cases} -\dfrac{4\pi}{3}\mathbf{P}_0, & r < R, \\ \left(\dfrac{4\pi}{3}R^3\right)\dfrac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{P}_0\cdot\hat{\mathbf{r}}) - \mathbf{P}_0\big], & R < r. \end{cases} \tag{18}$$

Compare this with the electric field due to a point dipole. Plot the electric field, both outside and inside the sphere.

(g) We have the constituent relation

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}. \tag{19}$$

Determine the expression for **D**. Plot **D**, both outside and inside the sphere. How is this different from the electric field.