Homework No. 04 (Fall 2018)

PHYS 520A: Electromagnetic Theory I

Due date: Monday, 2018 Sep 24, 4.00pm

1. (**40 points.**) Maxwell's equations with magnetic charge are

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_e, \tag{1a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \rho_m, \tag{1b}$$

$$-\boldsymbol{\nabla} \times \mathbf{E} - \frac{\partial}{\partial t}\mathbf{B} = \mathbf{J}_m, \tag{1c}$$

$$\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{J}_e, \tag{1d}$$

where

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{2a}$$

$$\mathbf{B} = \mu_0 \mathbf{H}. \tag{2b}$$

The Lorentz force, in SI units, is

$$\mathbf{F} = q_e \mathbf{E} + q_e \mathbf{v} \times \mathbf{B} + q_m \mathbf{H} - q_m \mathbf{v} \times \mathbf{D}. \tag{3}$$

We have

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}. \tag{4}$$

(a) Let us define

$$\mathbf{F} = \mathbf{E} + i\mathbf{H}, \qquad \rho = \rho_e + i\rho_m, \tag{5a}$$

$$\mathbf{G} = \mathbf{D} + i\mathbf{B}, \qquad \mathbf{J} = \mathbf{J}_e + i\mathbf{J}_m, \tag{5b}$$

where $i = \sqrt{-1}$. Then, show that the Maxwell equations are

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \rho, \tag{6a}$$

$$-i\boldsymbol{\nabla} \times \mathbf{F} - \frac{\partial \mathbf{G}}{\partial t} = \mathbf{J}. \tag{6b}$$

(b) Further, define

$$q = q_e + i q_m. \tag{7}$$

Show that, in terms of the complex conjugate $q^* = q_e - i q_m$,

$$q^* \mathbf{F} = (q_e \mathbf{E} + q_m \mathbf{H}) + i(q_e \mathbf{H} - q_m \mathbf{E}), \tag{8a}$$

$$q^* \mathbf{v} \times \mathbf{G} = (q_e \mathbf{v} \times \mathbf{D} + q_m \mathbf{v} \times \mathbf{B}) + i(q_e \mathbf{v} \times \mathbf{B} - q_m \mathbf{v} \times \mathbf{D}). \tag{8b}$$

Thus, write the Lorentz force in the presence of magnetic charge to be

$$\mathbf{F} = \mathrm{Re}\big[q^* \mathbf{F}\big] + \mathrm{Im}\big[q^* \mathbf{v} \times \mathbf{G}\big]. \tag{9}$$

(c) Consider the transformations

$$\mathbf{G} \to \mathbf{G}' = e^{-i\phi}\mathbf{G}, \qquad \rho \to \rho' = e^{-i\phi}\rho, \qquad q \to q' = e^{-i\phi}q, \tag{10a}$$

$$\mathbf{F} \to \mathbf{F}' = e^{-i\phi}\mathbf{F}, \qquad \mathbf{J} \to \mathbf{J}' = e^{-i\phi}\mathbf{J}. \tag{10b}$$

Show that the Maxwell equations do not change under these transformations if ϕ is uniform in space and time.

(d) If ϕ is not uniform show that the Maxwell equations transform into

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \rho + \rho^{\text{eff}}, \tag{11a}$$

$$-i\boldsymbol{\nabla} \times \mathbf{F} - \frac{\partial \mathbf{G}}{\partial t} = \mathbf{J} + \mathbf{J}^{\text{eff}}, \tag{11b}$$

where

$$\rho^{\text{eff}} = i(\boldsymbol{\nabla}\phi) \cdot \mathbf{G}, \tag{12a}$$

$$\mathbf{J}^{\text{eff}} = (\boldsymbol{\nabla}\phi) \times \mathbf{F} - i\left(\frac{\partial \phi}{\partial t}\right)\mathbf{G}. \tag{12b}$$

The real part of ρ^{eff} and $\mathbf{J}^{\text{eff}}$ are distinguishing features of a topological insulator.

2. (**40 points.**) Consider the motion of a particle with electric charge q_e and mass m in the field of a stationary particle with magnetic charge q_m. Reference: I. R. Lapidus and J. L. Pietenpol, Classical interaction of an electric charge with a magnetic charge, Am. J. Phys. **28** (1960) 17.

(a) Show that the magnetic field of a particle with magnetic charge is

$$\mathbf{H} = \frac{q_m}{4\pi\mu_0}\frac{\mathbf{r}}{r^3}. \tag{13}$$

(b) Using the Lorentz equation show that the equation of motion for the electric charge is

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\alpha}{m}\frac{d\mathbf{r}}{dt} \times \frac{\mathbf{r}}{r^3}, \qquad \alpha = \frac{q_e q_m}{4\pi}. \tag{14}$$

(c) Show that the kinetic energy

$$K = \frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 \tag{15}$$

is a constant of motion.
Hint: Take the scalar product of the equation of motion with velocity.

(d) Show that

$$\mathbf{L} = m\mathbf{r} \times \frac{d\mathbf{r}}{dt} \tag{16}$$

is not a constant of motion. However, show that $\mathbf{L}^2$ is a constant of motion. Further, show that

$$(\mathbf{L} - \alpha\hat{\mathbf{r}}) \tag{17}$$

is a constant of motion.

(e) Starting from the equation of motion show that

$$\mathbf{r} \cdot \frac{d^2\mathbf{r}}{dt^2} = 0. \tag{18}$$

Thus, derive

$$\frac{1}{2}\frac{d^2}{dt^2} r^2 = v^2, \tag{19}$$

where v is magnitude of velocity. Thus, the orbit is described by

$$r = \sqrt{(v^2 t + c)t + b^2}. \tag{20}$$

Thus, conclude that this motion does not permit bound states.

(f) Show that $\mathbf{L}^2$ is a constant of motion.

(g) Show that

$$\frac{1}{2}mv^2 + \frac{L^2 - \alpha^2}{2mr^2} \tag{21}$$

is a constant of motion.

(h) Show that $-\alpha\hat{\mathbf{r}}$ and $\mathbf{L}$ constitute two perpendicular sides of a right angled triangle, with the hypotenuse given by $(\mathbf{L} - \alpha\hat{\mathbf{r}})$. That is,

$$\hat{\mathbf{r}} \cdot \mathbf{L} = 0, \quad \hat{\mathbf{r}} \cdot (\mathbf{L} - \alpha\hat{\mathbf{r}}) = -\alpha, \quad \mathbf{L} \cdot (\mathbf{L} - \alpha\hat{\mathbf{r}}) = L^2. \tag{22}$$

Thus, conclude that the motion of the electric charge is confined to the surface of a cone whose axis is along $-(\mathbf{L} - \alpha\hat{\mathbf{r}})$ with cone angle θ given by

$$\cot\theta = \frac{\alpha}{L}. \tag{23}$$