Homework No. 05 (Fall 2018)

PHYS 520A: Electromagnetic Theory I

Due date: Friday, 2018 Oct 26, 4.00pm

1. (**20 points.**) Show that the effective charge density, ρ_{eff}, and the effective current density, $\mathbf{j}_{\text{eff}}$,

$$\rho_{\text{eff}} = -\boldsymbol{\nabla}\cdot\mathbf{P}, \tag{1}$$

$$\mathbf{j}_{\text{eff}} = \frac{\partial}{\partial t}\mathbf{P} + \boldsymbol{\nabla}\times\mathbf{M}, \tag{2}$$

satisfy the equation of charge conservation

$$\frac{\partial}{\partial t}\rho_{\text{eff}} + \boldsymbol{\nabla}\cdot\mathbf{j}_{\text{eff}} = 0. \tag{3}$$

2. (**60 points.**) Consider a solid cylinder of radius R and infinite length with uniform permanent polarization

$$\mathbf{P}(\mathbf{r},t) = \mathbf{P}_0\,\theta(R-\rho), \tag{4}$$

where $\rho^2 = x^2 + y^2$ and $\mathbf{P}_0$ is perpendicular to the axis of the cylinder. We shall find the electric potential and the electric field outside the cylinder.

(a) Show that the effective charge density is given by the expression

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} = \mathbf{P}_0\cdot\hat{\boldsymbol{\rho}}\,\delta(\rho-R). \tag{5}$$

(b) Beginning from

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho_{\text{eff}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{6}$$

after integrating by parts, and writing

$$\phi(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\mathbf{P}_0\cdot\boldsymbol{\nabla}\int d^3r'\,\frac{\theta(R-\rho')}{|\mathbf{r}-\mathbf{r}'|}, \tag{7}$$

show that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\theta(R-\rho')\,\frac{\mathbf{P}_0\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{8}$$

(c) Evaluate the integrals, z', ϕ', and ρ', to show that the electric potential outside the cylinder is given by

$$\phi(\mathbf{r}) = \frac{2\pi R^2}{4\pi\varepsilon_0}\,\frac{\mathbf{P_0}\cdot\boldsymbol{\rho}}{\rho^2}. \tag{9}$$

Hints:

i. In cylindrical coordinates we have

$$\boldsymbol{\rho} = \rho\cos\phi\,\hat{\mathbf{i}} + \rho\sin\phi\,\hat{\mathbf{j}}, \qquad \mathbf{r} = \boldsymbol{\rho} + z\,\hat{\mathbf{k}}, \tag{10a}$$

$$\boldsymbol{\rho}' = \rho'\cos\phi'\,\hat{\mathbf{i}} + \rho'\sin\phi'\,\hat{\mathbf{j}} \qquad \mathbf{r}' = \boldsymbol{\rho}' + z'\,\hat{\mathbf{k}}, \tag{10b}$$

$$\mathbf{P}_0 = P_0\cos\alpha\,\hat{\mathbf{i}} + P_0\sin\alpha\,\hat{\mathbf{j}}. \tag{10c}$$

Thus,

$$|\mathbf{r} - \mathbf{r}'|^2 = (z - z')^2 + |\boldsymbol{\rho} - \boldsymbol{\rho}'|^2 \tag{11}$$

and

$$\mathbf{P}_0 \cdot (\mathbf{r} - \mathbf{r}') = \mathbf{P}_0 \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}'). \tag{12}$$

ii. Complete the z' integral using

$$\int_{-\infty}^{\infty} \frac{dz}{(z^2 + a^2)^{\frac{3}{2}}} = \frac{2}{a^2} \tag{13}$$

to obtain the result

$$\phi(\mathbf{r}) = \frac{2}{4\pi\varepsilon_0} \int d^2\rho'\, \theta(R - \rho')\, \frac{\mathbf{P}_0 \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|^2}. \tag{14}$$

where $d^2\rho' = \rho' d\rho' d\phi'$.

iii. Choose $\phi = 0$. Then, complete the ϕ' integrals using

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{1}{(1 \pm a\cos\phi)} = \frac{1}{\sqrt{1 - a^2}}, \quad |a| < 1, \tag{15}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{\cos\phi}{(1 \pm a\cos\phi)} = \pm\frac{1}{a} \mp \frac{1}{a\sqrt{1 - a^2}}, \quad |a| < 1, \tag{16}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{\sin\phi}{(1 \pm a\cos\phi)} = 0, \quad |a| < 1. \tag{17}$$

iv. Collect (and complete cancellations) before completing the ρ' integral. The divergence iassociated to $\rho \to R$ cancels for $\rho' < R < \rho$.

(d) Evaluate the gradient of the electric potential to show that the electric field outside the cylinder is given by

$$\mathbf{E}(\mathbf{r}) = \frac{2\pi R^2}{4\pi\varepsilon_0} \frac{1}{\rho^2} \Big[2(\mathbf{P_0} \cdot \hat{\boldsymbol{\rho}})\hat{\boldsymbol{\rho}} - \mathbf{P}_0 \Big]. \tag{18}$$