Homework No. 07 (Fall 2018)

PHYS 520A: Electromagnetic Theory I

Due date: Monday, 2018 Nov 26, 4.00pm

1. (**50 points.**) Consider the differential equation

$$\left[-\frac{\partial}{\partial z}\varepsilon(z)\frac{\partial}{\partial z}+\varepsilon(z)k_\perp^2\right]g_\varepsilon(z,z')=\delta(z-z'), \tag{1}$$

for the case

$$\varepsilon(z)=\begin{cases}\varepsilon_2, & z<0,\\ \varepsilon_1, & 0<z,\end{cases} \tag{2}$$

satisfying the boundary conditions

$$g_\varepsilon(-\infty,z')=0, \tag{3a}$$

$$g_\varepsilon(+\infty,z')=0. \tag{3b}$$

(a) Verify, by integrating Eq. (1) around $z=z'$, that the Green function satisfies the continuity conditions

$$g_\varepsilon(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta}=0, \tag{4a}$$

$$\varepsilon(z)\frac{\partial}{\partial z}g_\varepsilon(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta}=-1. \tag{4b}$$

(b) Verify, by integrating Eq. (1) around $z=0$, that the Green function satisfies the continuity conditions

$$g_\varepsilon(z,z')\Big|_{z=0-\delta}^{z=0+\delta}=0, \tag{5a}$$

$$\varepsilon(z)\frac{\partial}{\partial z}g_\varepsilon(z,z')\Big|_{z=0-\delta}^{z=0+\delta}=0. \tag{5b}$$

(c) For $z'<0$, construct the solution in the form

$$g_\varepsilon(z,z')=\begin{cases}A_1e^{k_\perp z}+B_1e^{-k_\perp z}, & z<z'<0,\\ C_1e^{k_\perp z}+D_1e^{-k_\perp z}, & z'<z<0,\\ E_1e^{k_\perp z}+F_1e^{-k_\perp z}, & z'<0<z.\end{cases} \tag{6}$$

Determine the constants using the boundary conditions and continuity conditions.

(d) For $0 < z'$, construct the solution in the form

$$g_\varepsilon(z,z') = \begin{cases} A_2 e^{k_\perp z} + B_2 e^{-k_\perp z}, & z < 0 < z', \\ C_2 e^{k_\perp z} + D_2 e^{-k_\perp z}, & 0 < z < z', \\ E_2 e^{k_\perp z} + F_2 e^{-k_\perp z}, & 0 < z' < z. \end{cases} \tag{7}$$

Determine the constants using the boundary conditions and continuity conditions.

(e) Thus, find the solution

$$g_\varepsilon(z,z') = \begin{cases} \dfrac{1}{\varepsilon_2}\dfrac{1}{2k_\perp} e^{-k_\perp|z-z'|} + \dfrac{1}{\varepsilon_2}\dfrac{1}{2k_\perp}\left(\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right) e^{-k_\perp|z|} e^{-k_\perp|z'|}, & z' < 0, \\ \\ \dfrac{1}{\varepsilon_1}\dfrac{1}{2k_\perp} e^{-k_\perp|z-z'|} + \dfrac{1}{\varepsilon_1}\dfrac{1}{2k_\perp}\left(\dfrac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right) e^{-k_\perp|z|} e^{-k_\perp|z'|}, & 0 < z'. \end{cases} \tag{8}$$