# Homework No. 03 (2019 Spring)

## PHYS 301: Theoretical Methods in Physics

Due date: Monday, 2019 Feb 4, 9:00 AM, in class

1. (**10 points.**) (No submission needed for this part.) Self-study keywords: Cauchy-Riemann conditions, Cauchy's integral formula, Residue theorem, Laurant series.

2. (**30 points.**) Analytic functions are significantly constrained, in that they have to satisfy the Cauchy-Riemann conditions. These conditions are necessary (but not sufficient) for a function of a complex variable to be analytic (differentiable). Check if the following functions satisfy the Cauchy-Riemann conditions.

$$f(z) = z^3, \tag{1a}$$
$$f(z) = |z|, \tag{1b}$$
$$f(z) = e^{iz}, \tag{1c}$$
$$f(z) = \ln z. \tag{1d}$$

3. (**10 points.**) We constructed the Cauchy-Riemann conditions such that a unique first derivative exists. Show that Cauchy-Riemann conditions guarentee the existance of a unique second derivative and all higher derivatives.

4. (**10 points.**) Evalauate the following contour integrals. In the following the contour $c$ is a unit circle going counterclockwise with center at the complex number $a$.

$$I(a) = \frac{1}{2\pi i} \oint_c dz \frac{(z^5 + 1)}{(z - a)}, \tag{2a}$$
$$I(a) = \frac{1}{2\pi i} \oint_c dz \frac{e^{iz}}{(z - a)}. \tag{2b}$$

5. (**20 points.**) Consider the integral

$$I(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 + a\cos\theta)}, \tag{3}$$

where $a$ is complex. Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{4}$$

and express the integral as a contour integral,

$$I(a) = \frac{1}{2\pi i} \frac{2}{a} \oint_c \frac{dz}{\left(z^2 + \frac{2}{a} z + 1\right)}, \tag{5}$$

where the contour $c$ is along the unit circle going counterclockwise. Show that

$$z^2 + \frac{2}{a}z + 1 = (z - r_+)(z - r_-), \tag{6}$$

where

$$r_\pm = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}. \tag{7}$$

Using residue theorem evaluate $I(a)$ for $|\mathrm{Re}(a)| < 1$ and $\mathrm{Im}(a) = 0$.