Homework No. 08A (2019 Spring)

PHYS 301: Theoretical Methods in Physics

Due date: None. Practice on cylindrical polar coordinates.

1. (**10 points.**) In cylindrical polar coordinates a point is coordinated by the intersection of family of right circular cylinders, half-planes, and planes, given by

$$\rho = \sqrt{x^2 + y^2}, \tag{1a}$$

$$\phi = \tan^{-1} \frac{y}{x}, \tag{1b}$$

$$z = z, \tag{1c}$$

respectively. Show that the gradient of these surfaces are given by

$$\boldsymbol{\nabla}\rho = \hat{\boldsymbol{\rho}}, \qquad \hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2a}$$

$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\phi}}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2b}$$

$$\boldsymbol{\nabla}z = \hat{\mathbf{z}}, \qquad \hat{\mathbf{z}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{2c}$$

which are normal to the respective surfaces. Sketch the surfaces and the corresponding normal vectors. This illustrates that $\boldsymbol{\nabla}$(surface) is a vector (field) normal to the surface.

2. (**10 points.**) Using the gradient operator in cylindrical polar coordinates,

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{3}$$

evaluate the following

$$\frac{\partial}{\partial\rho}\hat{\boldsymbol{\rho}} = 0, \qquad \frac{\partial}{\partial\rho}\hat{\boldsymbol{\phi}} = 0, \qquad \frac{\partial}{\partial\rho}\hat{\mathbf{z}} = 0, \tag{4a}$$

$$\frac{\partial}{\partial\phi}\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}, \qquad \frac{\partial}{\partial\phi}\hat{\mathbf{z}} = 0, \tag{4b}$$

$$\frac{\partial}{\partial z}\hat{\boldsymbol{\rho}} = 0, \qquad \frac{\partial}{\partial z}\hat{\boldsymbol{\phi}} = 0, \qquad \frac{\partial}{\partial z}\hat{\mathbf{z}} = 0. \tag{4c}$$

Visualize the above variational statements graphically.

3. (**10 points.**) Evaluate the following divergence of vector fields.

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\rho}}, \quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}}, \quad \boldsymbol{\nabla}\cdot\hat{\mathbf{z}}, \tag{5a}$$

$$\boldsymbol{\nabla}\cdot(\rho^2\hat{\boldsymbol{\rho}}), \quad \boldsymbol{\nabla}\cdot(\rho^2\hat{\boldsymbol{\phi}}), \quad \boldsymbol{\nabla}\cdot(\rho^2\hat{\mathbf{z}}), \tag{5b}$$

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{z}}}{\rho}\right). \tag{5c}$$

Draw the vector fields. Visualize and interpret the action of the divergence operator. Which of the above are divergenceless.

4. (**10 points.**) Evaluate the following curl of vector fields.

$$\boldsymbol{\nabla} \times \hat{\boldsymbol{\rho}}, \quad \boldsymbol{\nabla} \times \hat{\boldsymbol{\phi}}, \quad \boldsymbol{\nabla} \times \hat{\mathbf{z}}, \tag{6a}$$

$$\boldsymbol{\nabla} \times (\rho^2 \hat{\boldsymbol{\rho}}), \quad \boldsymbol{\nabla} \times (\rho^2 \hat{\boldsymbol{\phi}}), \quad \boldsymbol{\nabla} \times (\rho^2 \hat{\mathbf{z}}), \tag{6b}$$

$$\boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho} \right), \quad \boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho} \right), \quad \boldsymbol{\nabla} \times \left(\frac{\hat{\mathbf{z}}}{\rho} \right). \tag{6c}$$

Draw the vector fields. Visualize and interpret the action of the curl operator. Which of the above are curl free.